USES OF HISTORY FOR THE LEARNING OF AND ABOUT MATHEMATICS

Towards a theoretical framework for integrating history of mathematics in mathematics education

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ABSTRACT

The purpose of the present paper is to present a theoretical framework for analyzing, criticizing and orienting designs and implementations of history of mathematics in mathematics education in order to address the questions of how integrating history of mathematics benefits students' learning of mathematics and how uses of historical elements to support students' learning of mathematics develop students' historical awareness. To address the second question, a multiple perspective approach to history of practices of mathematics is introduced together with a set of concepts that can be used to identify and articulate different forms of people's uses of history. To address the first question uses of history and history of mathematics are linked to a competence based conception of mathematics education and Sfard's theory of mathematics as a discourse. To illustrate the framework and how it can be used, two examples, one from a university master's program and one from a Danish high school, of integrating history into mathematics education are presented and analyzed.

1 introduction

Despite a range of well known arguments[1] for integrating history in mathematics classrooms, and the inclusion of history in the national mathematics curriculum in some countries,[2] history does not play a significant role in general mathematics education. This might seem strange for someone from the outside, considering that mathematics has a history that goes more than 5000 years back, so the past provides a huge reservoir of authentic mathematical texts and activities, and why not learn from the masters?[3] However, as every one knows who has tried it, it is not so straightforward to integrate history in mathematics teaching and learning.[4] On one hand there is the question of how integrating history of mathematics benefits students' learning of mathematics, and on the other hand when

[1]See e.g. Beckman (2009), Fauvel (1991a, 1991b), Fauvel and van Maanen (2000). See also the review article by Jankvist (2009).

[2]Examples are Denmark and France.

[3]David Pengelley and Reinhard Laubenbacher have developed several courses where they teach with original historical sources. These are described on their homepage `http://sofia.nmsu.edu/~history/`. They have published several papers and books explaining their approach, see e.g. Laubenbacher and Pengelley (1996).

[4]See e.g. Siu (2007).

historical elements are used to support students' learning of mathematical concepts, theories or techniques, or to humanize mathematics, there is the question of in what sense such implementations develop students' historical awareness, or convey a picture of mathematics as the epitome of timeless truth and mathematical objects as ideal timeless entities (see e.g. Otte (2007) and Epple (2011)).[5]

In the present paper I will focus on these two issues. To deal with the second issue, I will in section 2) introduce notions from research in people's uses of history and from the academic discipline of history of mathematics. Recent research has shown that people use history in many different contexts, with different approaches and for different purposes, i.e. we attach several, partly different meanings to history. The task is not to announce one approach as the right one and discard the others, but to unfold the differences between the various ways in which history is being used and understood. The challenge is not to dissolve the complexity but to explore it; to clarify how history is (can be) understood and for what purposes it is or can be used, in order to capture some of the multifaceted ways in which history can benefit students' learning of and about mathematics.

To deal with the first issue, uses of the past and history of mathematics needs to be linked to theories from didactics that connects to conceptions of mathematics education and to learning of mathematics. This will be done in section 3) and section 4), respectively. Ideas from didactics of mathematics are introduced to discuss and analyze how and in what sense different approaches to history can benefit teaching and learning of mathematics. In section 3) Mogens Niss' proposal for a competence based understanding of mathematics education that addresses the question of what it means to master mathematics is introduced.[6] In section 4) Anna Sfard's theory of mathematics as a discourse is presented and I will argue, that history of mathematics can function at the core of what it means to learn mathematics.[7]

Together, these theories and notions will span a theoretical framework that can be used to place, analyze and criticize implementations of historical elements in mathematics classrooms to understand how history is used and in what sense it can benefit students' learning of mathematics. It can also be used to orient designs of implementations such that learning goals and teaching intentions can be made clearer and targeted – whether these goals are directed towards the learning of mathematics or of history of mathematics. The framework allows teachers to make informed and reflected choices about how and for what purpose(s) historical elements can enter mathematics classrooms.

To illustrate the theoretical framework and how it can be used, two examples of integrating history into mathematics education will be presented and analyzed in section 5). The first example is a report from a project work carried out by a group of students in a university master's programme in mathematics in Denmark. The second example is an experimental teaching course on implementing problem oriented project work in history of mathematics that was carried out in a Danish high school.

2 A multiple perspective approach to history of mathematics and different uses of history

The notion of a "multiple perspective" approach to history is borrowed from the Danish historian Eric Bernard Jensen's (2003, 16–17) writings about historiography. The multiple perspectives enter,

[5]For discussions of these two issues, see also Fried (2001, 2007).
[6]See Niss (2004) and Niss and Højgaard (2011).
[7]See Sfard (2008).

because the underlying premise is that people are understood as being shaped by history and being shapers of history. History is studied from perspective(s) of the historical actors, paying attention to these actors' intentions and motivations, as well as to intended and unintended consequences of their actions. It is an action-oriented conception of history were people, their projects and their actions are taken as point of departure for historical investigations to achieve a historical-social understanding of how people have thought and acted at different times and in different cultures.

Such an approach to history can be adapted to history of mathematics, if we think of mathematics as a cultural and historical product of knowledge that is produced by human intellectual activities. The knowledge that is produced by a mathematician, or group of mathematicians, at a certain time in history depends on the knowledge and mathematical culture available for these mathematicians and it (might) shape or define guidelines for further developments of mathematical knowledge. In this sense, such an action-oriented perception of history of mathematics can be pursued, where the historian study the history of mathematics from perspectives of past mathematicians, their projects and motivations, situated in certain contexts, at specific places, at certain times, and under particular historical circumstances in order to understand and explain historical processes in the development of mathematics. In the academic profession of history of mathematics, contextualized historical investigations of this kind are undertaken. One approach it to study concrete episodes of production of mathematical knowledge within the "work place" of the involved (past) mathematicians, studying the development of these mathematicians' production of mathematics from their practice(s) of mathematics, trying to follow the development of these mathematicians' ideas and techniques.[8]

Besides the perspectives of the historical actors, the perspective of the historian also needs to be taken into account. A historian's research inquiry is always guided by some questions, problems, or wonderings that she/he wants to answer, solve or understand. Hence, the choice of perspective(s) is determined in a dialectic process between the historian's perspective(s), i.e. what she/he wants to understand regarding the historical episode in question, and the historical actors' perspectives as they unfold during the research process.

The strength of such a multiple perspective approach where the development of mathematics is studied from different points of observation is that the historical analyzes are attached to concrete episodes of mathematical research and research practices from where relations and connections can be unfolded and explored. The perspectives can be of different kinds. In some instances the historian might be interested in e.g. how other disciplines influenced the development of pieces of mathematics, or how and why techniques of proofs changed, or if and how applications of mathematics influenced its developments etc., asking questions such as why mathematicians introduced specific definitions and concepts, which particular problems did they work on, what techniques did they use and why, how did mathematical objects emerge and develop.[9] These kinds of perspectives and historical questions regarding mathematical research practices relate to the content and inner core of mathematics, and consequently, such a multiple perspective approach to history of mathematics studied from practices of mathematics has the potential to play a significant role for the learning of mathematics.

This approach to history of mathematics comes from the academic discipline i.e. how professional

[8]See e.g. Leo Corry's introduction (Corry, 2004) as well as the rest of the papers published in *Science in Context*, 17($\frac{1}{2}$), 2004. See also Epple (2000), Kjeldsen (2004), Kjeldsen and Carter (2012) to name just a few where also further references can be found.

[9]See Epple (2004).

historians of mathematics think and practice history. Research into people's uses of history has shown that such an academic approach to history is jut one of many approaches. It has shown that people's historical awareness is formed in many different contexts, that people use history in many different connections and for many different purposes, e.g. in movies, when we travel, in family histories, in computer games, in school subjects, in museums, in memorial places and landmarks.[10]

Jensen has written about people's conception and uses of history. In this context, he defines history as follows: "when a person or a group of people is interested in something from the past and uses their knowledge about it for some purpose" (Jensen 2010, 39). When history is viewed in this broader perspective it becomes a complex concept—an umbrella term for a collection of related forms of knowledge and practices that people uses in their life. The task is not to identify one form of history as the right one, but to reveal similarities and differences in approaches and ways in which history is understood and used.

For this purpose, Jensen (2010, 40–57; 141; 145) has introduced four pairs of concepts, which he uses to identify and articulate different forms of people's uses of history. They are: (1) lay history and professional history; (2) pragmatic history and scholarly history; (3) action history and observer history; (4) identity concrete and identity neutral history.[11] These concepts can be used as guideposts when we want to understand and analyze our own and other's conception and uses of history. They address different aspects: methodological aspects of research in history, history as an academic field of research and an ʻevery day use' of history, and the intentions of specific uses of history. Hence, they do not mutually exclude one another, they overlap, and they can be present in various degrees in concrete uses of history.

Lay history and *professional history* distinguishes between every-day (life world) uses and professional uses of history i.e. it is about differences in the context in which history is used. Lay people's (i.e. non-professional historians') uses of history have become an object of research within the last decades. According to Schörken (1981), professional historians consider lay history to be naïve and lay people think of professional history as lifeless and distant from the real world. Many mathematicians including mathematics teachers read and use past mathematical texts for research, in teaching and out of sheer interests for its history. History of mathematics is also an academic discipline with its own research programmes, educational programmes, academic degrees and prices, journals, international conferences etc. Hence, the distinction between lay history and professional history makes sense when it comes to conceptions and uses of history of mathematics and some of the historiographical debates exhibit this distinction. To give just one example, the historian of mathematics Grattan-Guinness (2004, 163) complains that mathematicians are not sympathetic to history (as professional historians of mathematics conceive of it) because "their normal attention to history is concerned with heritage: that is, how did we get there? Old results are modernized in order to show their current place; but the historical context is ignored and thereby often distorted. By contrast, the historian is concerned with what happened in the past, whatever the modern situation is." What Grattan-Guiness draws attention to in this quote is a difference that is one of the characteristics between lay persons' and professional historians' concerns of and with the past.

Pragmatic history is history studied from a kind of utility perspective. This is the case when history is conceived of as "the master of life" so to speak when we think, we can learn from history's

[10] See e.g. Ashton and Kean (2009), Eriksen and Sigurdsson (2009), Jensen (2010).

[11] My translation into English.

mistakes that history can teach us better ways to live our lives, the *historia magistra vitae* conception of history. A pragmatic historian will try to make history relevant in a contemporary context. Many professional historians now a day dissociate themselves from a pragmatic conception of history which they think inhibits our understanding of history epistemologically. They favour a *scholarly approach to history*[12] where they maintain a critical distance to the past and emphasise differences between now and then. History is about gaining insights into and understanding the past on its own terms. The multiple perspective approach to history of mathematics as described above would be characterized as a scholarly approach to history. The distinction between pragmatic and scholarly history overlaps with the distinction between lay history and professional history in the sense that lay people often have a pragmatic conception of history, whereas many professional historians now a days have a scholarly approach, but there are situations were lay history is guided by scholarly interests and the pragmatic approach to history has been a tradition within academic, professional history with the scholarly approach being the dominant one from the mid 19^{th} century (Jensen 2010, 48–52).

The notions *Action history* and *observer history* are used to distinguish between whether people look at a past episode retrospectively or in a forward-looking perspective. It is about people's position regarding a past episode. The term action history is used to characterize approaches to history where the past is used to orient one self and/or act in a present context. Jensen calls this an intervening use of history. In contrast, history can be used in a retrospective perspective with an enlightening purpose, in such cases Jensen (2010, 41) talks about *observer history*. As mentioned above, these concepts do not exclude one another e.g. an observer history can be contained in an action history. Jensen (2011, 8) gives the example of a professional historian who uses a scholarly approach to understand something from the past (e.g. a war) in order to spread information and enlighten people in the present (about the relation to the country of the war).

History can be used in an intervening sense to form people's identity and in such cases Jensen talks about an identity concrete presentation of history. What is considered to be identity concrete or identity neutral history writing depends on culture and time – a history writing that is considered to be identity neutral in one culture might not be considered to be neutral by another culture, and what is considered to be an identity neutral history writing at one point in time might be considered to be identity concrete at another point in time (Jensen 2010, 52-57).

Besides the approaches to history covered in these four pairs of concepts, Jensen also includes the so-called '*living history*' concept as a playful approach to history. This form of history, where people actively participate in historical scenes and experience life from reconstructed specific historical periods and settings (e.g. a late 14^{th} century market town) is a way of using history to help participants develop historical awareness. According to Jensen (2010, 145), many people find the living history approach appealing, because the playful approach with its focus on developments of skills requires other learning strategies than the more intellectual approach that is used in much school teaching, where students learn from books.

These notions provide a set of glasses—a lens—through which we can identify, articulate and distinguish between different understandings and uses of history. Together with the multiple perspective approach to history of mathematics outlined above, they provide a theoretical framework that can be used to characterize, analyze and criticize uses and practices of history and implemen-

[12]This is my translation of the Danish word "lærd"—which mean to be a scholarly person.

tations of history in mathematics classrooms. They can also be used to orient designs and future implementations of history to clarify and target learning goals and teaching intentions.

In the next sections, history of mathematics will be linked to theories from didactics that connect to conceptions of mathematics education and to learning of mathematics, in order to discuss aspects of how and in what sense history can function in mathematics teaching and learning within these theories.

3 A competence based mathematics education—and the role of history

By a competence based view of mathematics education, I refer to the understanding of mathematics education as it has been developed in the Danish KOM-project (Niss 2004, Niss and Højgaard 2011). The project was initiated by the Danish Ministry of Education, and its understanding of mathematical competence form the basis for curriculum developments and descriptions in general mathematics education in Denmark. The objective of the project was to identify purposes and learning outcomes of mathematics education when the goal is to educate people to master mathematics.[13] Instead of a traditional curriculum of lists of concepts, subjects, techniques, results, etc., the project group identified eight main competencies and three kinds of second order competencies that were argued to span mathematical competence.[14]

The eight main competencies are divided into two groups: Four competencies that have to do with abilities to ask and answer questions in and with mathematics, and four competencies that regard abilities and familiarities with language and tools in mathematics, see Figure 1.

1. Thinking competency	5. Representing competency
2. Problem tackling competency	6. Symbol and formalism competency
3. Modeling competency	7. Communicationg competency
4. Reasoning competency	8. Aids and tools competency

Figure 1: The two groups of main competencies.

According to the Danish KOM-project, mathematics education should also provide students with philosophical and historical insights of mathematics to achieve a balanced picture of mathematics. The three second order competencies take care of that. They are (1) meta-issues of actual applications of mathematics in other subject and practice areas, (2) the historical development of mathematics in cultures and societies, its internal and external driving forces and interactions with other fields, and (3) the nature of mathematics as a discipline (Niss 2004).

The intentions of the KOM-project behind the second order competency of historical awareness and insights are in accordance with the multiple perspective approach to history of mathematics studied from its practice, as it is described above. Hence, the historical awareness that the KOM-project wants to develop in students corresponds to a scholarly perception of history. Such an approach to

[13]Some find the word "competence" non-appropriate because it labels students as being non-competent if they do not succeed in mathematics. The word competence was chosen because it focuses on abilities to cope with situations where mathematics plays or can play a role (Niss 2004, 182).

[14]A mathematical competency are developed and trained in relation to subject matters of mathematics. Ten subject areas are identified in the KOM-report as subject matters in which to develop and train mathematical competence in general education: The number domain, arithmetic, algebra, geometry, functions, calculus, probability theory, statistics, discrete mathematics and optimization (Niss and Højgaard 2011, 126-128).

history also has potential to train and develop (some) of students' main mathematical competencies. This can be achieved by using the multiple perspective approach to history of mathematical practices in mathematics education on a small scale, by focusing on a limited amount of carefully chosen perspectives that address issues in concrete pieces of past mathematical activities, in order to have students become aware of and reflect upon e.g. research strategies or the function and nature of specific mathematical concepts, arguments, problems, methods, results etc. from the historical episode in question. In section 5) we will see an example where students, through such an approach to history, developed historical awareness in the sense of the KOM-project in a way that also invoked and trained (some) of the students' main mathematical competencies. This is an example where observer history is contained in action history.

4 Can history function at the core of what it means to learn mathematics?

In the competence based understanding of mathematics education, history as such is part of mathematics education through the second order competencies. Students' mathematical competencies can be invoked and trained in the process of developing their second order competency of historical awareness, but history is not essential for developing students' first order mathematical competencies. Therefore, in this section I will address the question whether history can function at the core of what it means to learn mathematics?

Here I will draw on Sfard's (2008) theory of *Thinking as Communicating*. The framework of mathematical competence presented above deals with how we can think of mathematics education in terms of what should come out of mathematics education, namely people who posses mathematical competence to some degree.[15] Sfard (2008) is concerned with human thinking in general and mathematical thinking in particular.

Sfard views mathematics as a discourse where discourse "refers to the totality of communicative activities, as practiced by a given community" (Sfard 2000, 160). Learning mathematics then means to become a participant in the discourse. Discourse denotes human activity, and the discursive interpretation of learning emphasizes the social nature of intellectual activities. The activity of communicating is regulated by rules. Sfard distinguishes between two types of rules: object-level rules and metalevel rules. Object-level rules concern the content of the discourse. They are narratives about properties of mathematical objects. Metalevel rules are rules about the discourse itself. They are implicitly present and they govern "when to do what and how to do it." (Sfard 2008, 201–202). They

> "manifest their presence $\cdots$ in our ability to decide whether a given description can count as a proper mathematical definition, whether a given solution can be regarded as complete and satisfactory from a mathematical point of view, and whether the given argument can count as a final and definite confirmation of what is being claimed." (Sfard 2000, 167)

To become a participant in mathematics discourse, to learn mathematics, it is necessary to develop not only object-level rules but also proper metalevel rules. Hence, creating situations where metalevel

[15]In the KOM-report it is suggested that progression in an individual's mathematical competence is realised through its growth in three dimensions: degree of coverage, radius of action and technical level, (Niss and Højgaard 2011, 30).

rules are exhibited and made explicit objects of students' reflection is an essential aspect of mathematics teaching and learning – and it is with regard to this history of mathematics can function at the core of what it means to learn mathematics, because metalevel rules are contingent. These rules are not necessary. They develop and change over time. This means that these rules can be investigated at the object-level of history discourse.

According to Sfard, because of the contingency of metalevel rules, students are not likely to begin a metalevel change by themselves. This is most likely to happen if the learner becomes confronted with another discourse governed by metalevel rules that are different from the ones she or he has been acting in accordance with so far. Sfard have termed such an experience a *commognitive* conflict, and she defines it as "a situation in which different discursants are acting according to different metarules." (Sfard 2008, 256).

The scholarly multiple perspective approach to history of mathematics studied from its practice as outlined in section 2) views mathematics as a historical and cultural product of knowledge that develops because of human intellectual activities. Hence, there is no contradiction between this approach to history and a discursive view of mathematics. As we have argued in Kjeldsen and Blomhøj (2012), and as will be illustrated below, metalevel rules can be exhibited as explicit objects of reflection for students, by having students work with historians' tools on historical questions about the practice of mathematics in concrete mathematical episodes from the past. History provides a huge reservoir of authentic mathematical texts either published mathematical articles, correspondences between mathematicians, manuscripts for talks, and notes etc. Such sources can play the role as an "interlocutor". Students can examine them in their historical context and analyze the work of former mathematicians with respect to the way they formulated mathematical statements, the way they argued for their claims, their views on mathematics and so on. Hereby students can experience differences in meta-discursive rules between interlocutors (the historical text, themselves, their textbooks and/or their instructor). In this way metalevel rules can be revealed and made the object of students' reflections. Whether commognitive conflicts occur, depends on the chosen sources, and the metalevel rules that govern the students' own discourse. It is of course essential to use a scholarly approach to history, i.e. gaining insights into and understanding the past on its own terms. The so-called whig (or present-centredness) interpretation of history, where the readings and interpretations of historical sources are constrained by our modern conception of mathematics must be avoided.[16] If past mathematics is translated into modern mathematics and "old results are modernized", as was quoted from Grattan-Guiness in the description of lay history above, differences in discourse between the past and the present will (partly) disappear.

In the first example of the next section, some instances will be given, where metalevel rules were made into explicit objects of reflection for students through the students' work with sources from the past and historical investigations of an episode from the history of differential equations.[17]

[16]The term whig history comes from Butterfield (1931). See also Wilson and Ashplant (1988) and Schubring (2008).

[17]In Kjeldsen and Petersen (forthcoming) another example is presented where learning and teaching situations were designed using parts of the framework, with intention to make meta-discursive rules in mathematics objects of students' reflection and to detect students' metarules. Here the past was used with a deliberate intention of intervening. i.e. it is an example where observer history is contained in actor history.

5 Two examples: analysis of a student project and an experimental teaching course

In this section, two examples from teaching practice will be analysed and discussed within the framework developed and presented in section 2), 3) and 4). The first one is a project work conducted by a group of students in a university programme in mathematics and the second one is an experimental teaching course that was implemented and studied in a mathematics classroom in upper secondary school. Both of them function as examples of how concrete implementations of history can be analyzed to understand how history was used and in what sense it benefitted (or had the potential to benefit) students learning of mathematics.

5.1 Analysis of a student project work from a university master's programme in mathematics

Roskilde University is a reform university that was founded in 1972 in Roskilde, Denmark. The university implemented student centred, problem oriented and group organized project work as one of its main pedagogical principles. All students of the university, no matter which study programme they follow, participate in a group organized project work in every semester. The project work runs throughout the entire semester. At the end of the semester each group hands in a report of 50–100 pages in which they answer the problem formulation that has guided their project work. Besides the project work, students also follow regular courses. Course work and project work run in parallel each semester, and they each take up half a student's study load. A student has participated in 10 such projects when he or she receives his/her master's degree.[18] The project work analyzed below belongs to the first semester of the master's programme in mathematics.[19]The theme of the project is "mathematics as a discipline", and the requirement is that the students should work with a problem through which they will gain insights into the nature of mathematics and its "architecture" as a scientific discipline in a way that illustrates the historical developments of mathematics, its status and/or its place in society.

The project report in question was written by five students. It has the title *Physics' Influence on the Development of Differential Equations*. From their courses and project work in mathematical modelling in their bachelor studies, the students had experienced that differential equations play a central role in applications of mathematics in other sciences. They wanted to investigate how differential equations were developed and what motivated that development. They knew that during the last part of the 17^{th} century, mathematicians had begun to use infinitesimals to solve problems that were difficult to solve with classic geometry. Many of these problems were physical problems, and, as the students wrote in their report, physics is often mentioned in history of mathematics literature as an influential factor in the development of differential equations. The students were curious to find out how and in what sense physics had influenced the development of differential equations in the 17^{th} and 18^{th} century. Their historical investigations were guided by questions such as: "How did physics influ-

[18]Readers interested in the special problem oriented project work are referred to Kjeldsen and Blomhøj (2009), Blomhøj and Kjeldsen (2009), Salling Olesen and Højgaard Jensen (1999). See Niss (2001) for further information and discussions about the experiences with problem oriented student projects at Roskilde University.

[19]Before entering into the master's programme in mathematics, students have completed a three year interdisciplinary science of bachelor's programme where they have specialized in mathematics and one other subject.

ence the development of differential equations? Was it as problem generator? Did physics play a role in the formulation of differential equations as solutions to given problems? To what extent can the influence from physics be traced in the first systematization of the theory of differential equations?" (Paraphrased from Nielsen, Nørby, Mosegaard, Skjoldager and Zacho (2005, 8))

To answer these questions, the students studied this episode in history of mathematics from its practice from the perspective of how problems from another discipline (physics) influenced the development of mathematics,[20] how they entered into mathematicians' formulation of problems and the techniques they used to solve the problems. The students chose two cases: the catenary problem and the brachistochrone problem. The catenary problem is to find the shape of the curve formed by a flexible string that hangs freely between two fixed points. The brachistochrone problem is to find the path of fastest descend for a point that moves from one fixed point to another only influenced by gravity. They read and analyzed three selected original sources from the 1690s that dealt with the two cases: Johann Bernoulli's solution of the catenary problem and of the brachistochrone problem and Jakob Bernoulli's solution of the brachistochrone problem.

The students studied and interpreted the three sources within the mathematical discourse of the time, discussing them within the broader social and cultural context of the contemporary mathematical community. To mention just a few points: (1) the students discussed what was to be understood by a mathematician at that time, (2) they explained that the borders between disciplines were much looser than today and that mathematics and natural philosophy were much more intertwined, (3) they outlined how mathematical results were circulated (or not) within the mathematical community at the time, and emphasized the importance of competition which they linked to how mathematicians functioned in society, (4) they took into account the perspective of the actors, by discussing the content of the sources with respect to the Bernoulli brothers' intentions e.g. whether the purpose of the brothers' work was to solve the problems of the catenary and the brachistochrone or rather to investigate the effectiveness of infinitesimals as a new technique in mathematics in the 17^{th} century.

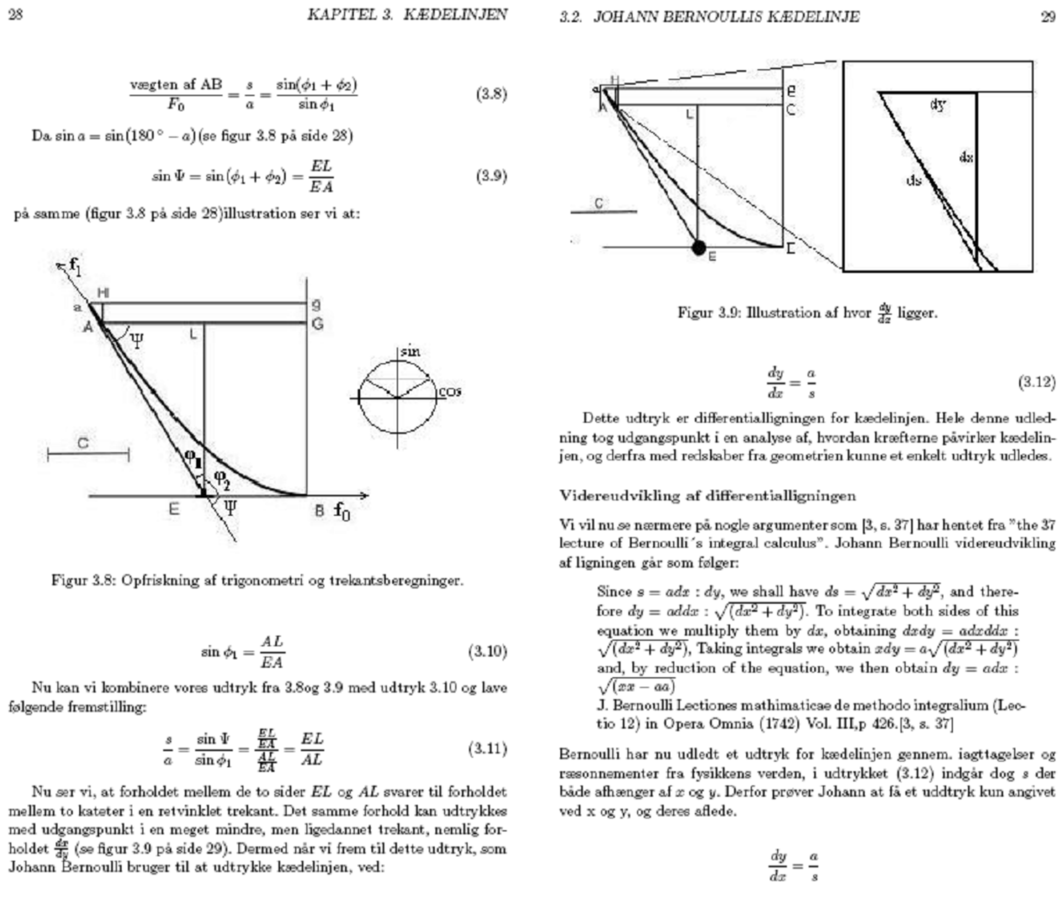
If we use the framework presented in section 2) to analyze this particular implementation of history into mathematics education to answer the second question that was raised in the introduction i.e. in what sense such an implementation develops students' historical awareness, we can conclude that for this particular implementation, the students had a scholarly approach to history with an enlightening purpose, i.e. observer history in Jensen's terminology.

In dealing with the mathematical content of the sources, the students made a detailed analysis of how the Bernoulli brothers derived the differential equations for the problems, how they formulated the equations and why they formulated them the way they did, how and with which methods they solved the equations. The students analyzed the sources with respect to what objects the Bernoulli brothers were investigating and which techniques they used to produce knowledge about the objects. All these issues are not only relevant for answering the students' historical questions, they are also relevant for the learning and understanding of differential equations. In the following I will analyze parts of the students' work within the framework presented in section 3) and 4) to answer the first question that was raised in the introduction, i.e. how integrating history of mathematics can benefit students' learning of mathematics. I will not go into all the details of the students' project work. In-

[20]The students realized in the course of the project work that physics and mathematics were not separated disciplines in the 17th century in the sense of how we consider them today, and that natural philosophy (as it was called) and mathematics were much more intertwined.

terested readers are referred to Kjeldsen (2011) for a discussion of mathematical competence, and to Kjeldsen and Blomhøj (2012) for a comprehensive analysis of the project work with respect to possibilities for meta-level learning.

In the catenary problem, Johann Bernoulli used five hypotheses from statics. In studying his treatment of the problem, the students had to mathematize these five hypotheses and to understand how Johann Bernoulli used them to describe the catenary. In working out this part of Bernoulli's text, the students' problem tackling competency, reasoning competency, representing competency, parts of their modelling competency[21] and their competency to handle symbols and formalism in mathematics were evoked and trained. In order to understand Bernoulli's mathematical representation of the catenary, they had a) to fill out many gaps themselves and derive intermediate results using arguments with similar triangles and from trigonometry, b) to introduce and understand the use of symbols, c) to mathematize the hypotheses. Figure 2 displays a couple of pages from the students' final report where they explain, how Bernoulli mathematized the hypotheses from statics and described the catenary.



28 KAPITEL 3. KÆDELINJEN

$$\frac{\text{vægten af AB}}{F_0} = \frac{s}{a} = \frac{\sin(\phi_1 + \phi_2)}{\sin \phi_1} \tag{3.8}$$

Da $\sin a = \sin(180° - a)$ (se figur 3.8 på side 28)

$$\sin \Psi = \sin(\phi_1 + \phi_2) = \frac{EL}{EA} \tag{3.9}$$

på samme (figur 3.8 på side 28)illustration ser vi at:

Figur 3.8: Opfriskning af trigonometri og trekantsberegninger.

$$\sin \phi_1 = \frac{AL}{EA} \tag{3.10}$$

Nu kan vi kombinere vores udtryk fra 3.8og 3.9 med udtryk 3.10 og lave følgende fremstilling:

$$\frac{s}{a} = \frac{\sin \Psi}{\sin \phi_1} = \frac{\frac{EL}{EA}}{\frac{AL}{EA}} = \frac{EL}{AL} \tag{3.11}$$

Nu ser vi, at forholdet mellem de to sider EL og AL svarer til forholdet mellem to kateter i en retvinklet trekant. Det samme forhold kan udtrykkes med udgangspunkt i en meget mindre, men ligedannet trekant, nemlig forholdet $\frac{dx}{dy}$ (se figur 3.9 på side 29). Dermed når vi frem til dette udtryk, som Johann Bernoulli bruger til at udtrykke kædelinjen, ved:

3.2. JOHANN BERNOULLIS KÆDELINJE 29

Figur 3.9: Illustration af hvor $\frac{dy}{dx}$ ligger.

$$\frac{dy}{dx} = \frac{a}{s} \tag{3.12}$$

Dette udtryk er differentialligningen for kædelinjen. Hele denne udledning tog udgangspunkt i en analyse af, hvordan kræfterne påvirker kædelinjen, og derfra med redskaber fra geometrien kunne et enkelt udtryk udledes.

Videreudvikling af differentialligningen

Vi vil nu se nærmere på nogle argumenter som [3, s. 37] har hentet fra "the 37 lecture of Bernoulli´s integral calculus". Johann Bernoulli videreudvikling af ligningen går som følger:

> Since $s = adx : dy$, we shall have $ds = \sqrt{dx^2 + dy^2}$, and therefore $dy = addx : \sqrt{(dx^2 + dy^2)}$. To integrate both sides of this equation we multiply them by dx, obtaining $dxdy = adxddx : \sqrt{(dx^2 + dy^2)}$, Taking integrals we obtain $xdy = a\sqrt{(dx^2 + dy^2)}$ and, by reduction of the equation, we then obtain $dy = adx : \sqrt{(xx - aa)}$
> J. Bernoulli Lectiones mathimaticae de methodo integralium (Lectio 12) in Opera Omnia (1742) Vol. III,p 426.[3, s. 37]

Bernoulli har nu udledt et udtryk for kædelinjen gennem. iagttagelser og ræsonnementer fra fysikkens verden, i udtrykket (3.12) indgår dog s der både afhænger af x og y. Derfor prøver Johann at få et uddtryk kun angivet ved x og y, og deres aflede.

$$\frac{dy}{dx} = \frac{a}{s}$$

Figure 2: Page 28 and 29 of the students' report. The text is in Danish except from the quote from the source on page 29.

Bernoullli described the infinitesimals dx and dy of the curve geometrically and he used the so-called infinitesimal triangle to derive an equation between dx and dy. This part of Bernoulli's text presented cognitive obstacles for the students. In order to understand Bernoulli's arguments, the students

[21]Modelling competency is understood in the sense of Blomhøj and Kjeldsen (2010).

had to read and understand the text within the mathematical discourse of the time which is difficult, because the point of departure for them was their own mathematical discourse, which was different from Bernoulli's. As explained in section 4) it is precisely the contingency of metalevel rules of mathematical discourse that is the reason, why history can serve as a means to reveal meta-discursive rules and make them objects of students' reflections. The way Bernoulli used geometry, statics and infinitesimals to derive an equation for the infinitesimals of the catenary is very different from the way it was introduced in the textbooks from which the students had learned about differential equations. Especially Bernoulli's use of the infinitesimal triangle went fundamentally against the $\epsilon-\delta-$conception of rigour that the students' had been brought up with in their first year analysis course. The right page (page 29) of figure 2 shows the students' explanation and discussion about the infinitesimal triangle.

In this part of the project work, the students' competencies to think and reason mathematically were trained and challenged in a new context provided by the history. The thoughts and reasoning presented in the historical sources were very different from the way in which issues of infinitesimals calculus are thought about and reasoned with in their analysis textbooks. For the students, the historical context provided an authentic piece of mathematics that could not be dealt with and understood on its own terms by using standard methods from their analysis textbooks. This is an instance where connections were created in the learning situation between the students' experiences with the involved mathematics from their textbook and their historical experience—an instance that challenged the students to use other aspects of their mathematical conceptions of infinitesimal calculus in new situations provided by the historical context. To be more concrete, the students' problems with understanding Bernoulli's way of reasoning with infinitesimals provoked situations where the students examined why Bernoulli's method worked in this particular case, and it initiated discussions between the students and the professor who supervised their project work about criteria for rigour and how such criteria are determined. These were instances where the students experienced that standards for rigour have changed over time, which is an example of the contingency of metalevel rules in mathematics discourse.

Another instance for metalevel learning occurred when the students had to understand Johann Bernoulli's solution of the catenary differential equation. Figure 3 is a picture of a page from the students' report that belongs to the section in the report where they explained and discussed Bernoulli's method.

Bernoulli constructed the solution geometrically. This is not how the students were used to solve differential equations and this part of their work initiated discussions between the group and their supervisor about conceptual aspects of what a solution to a differential equation means. They became aware that this, too, changes over time. It is a metalevel rule of mathematics discourse. The learning and teaching situations were these kind of discussions emerged were created in the process of the project work, guided by the students' efforts to read and understand the historical sources in order to find answers to their guiding questions. To illustrate how the students' "dialog" with the original source made them reflect upon these metalevel rules, I have translated the following two paragraphs from the students' report. In the first paragraph, the students investigate and discuss what counts as a valid argument, and in the second they reflect upon style of argumentation and generality in mathematics:

> "On page 26 we can follow how Johann Bernoulli transformed his physical knowledge into mathematics. Approximately half of Johann Bernoulli's account for the derivation of the differential

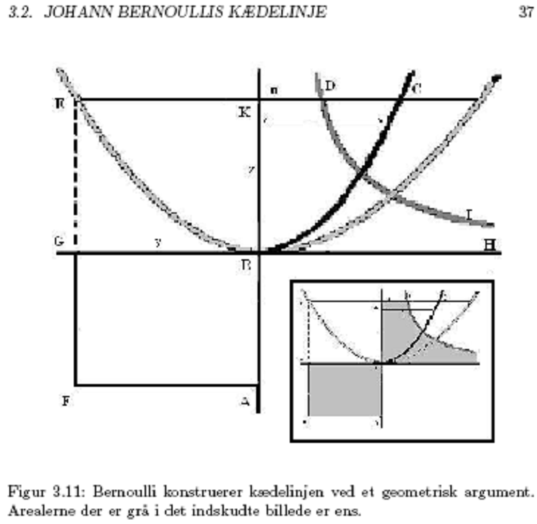


Figure 3: From page 37 of the students' report. Bernoulli's geometrical construction of the catenary. The figure with the two shadowed areas was constructed by the students to illustrate a comparison of areas that are involved in Bernoulli's construction.

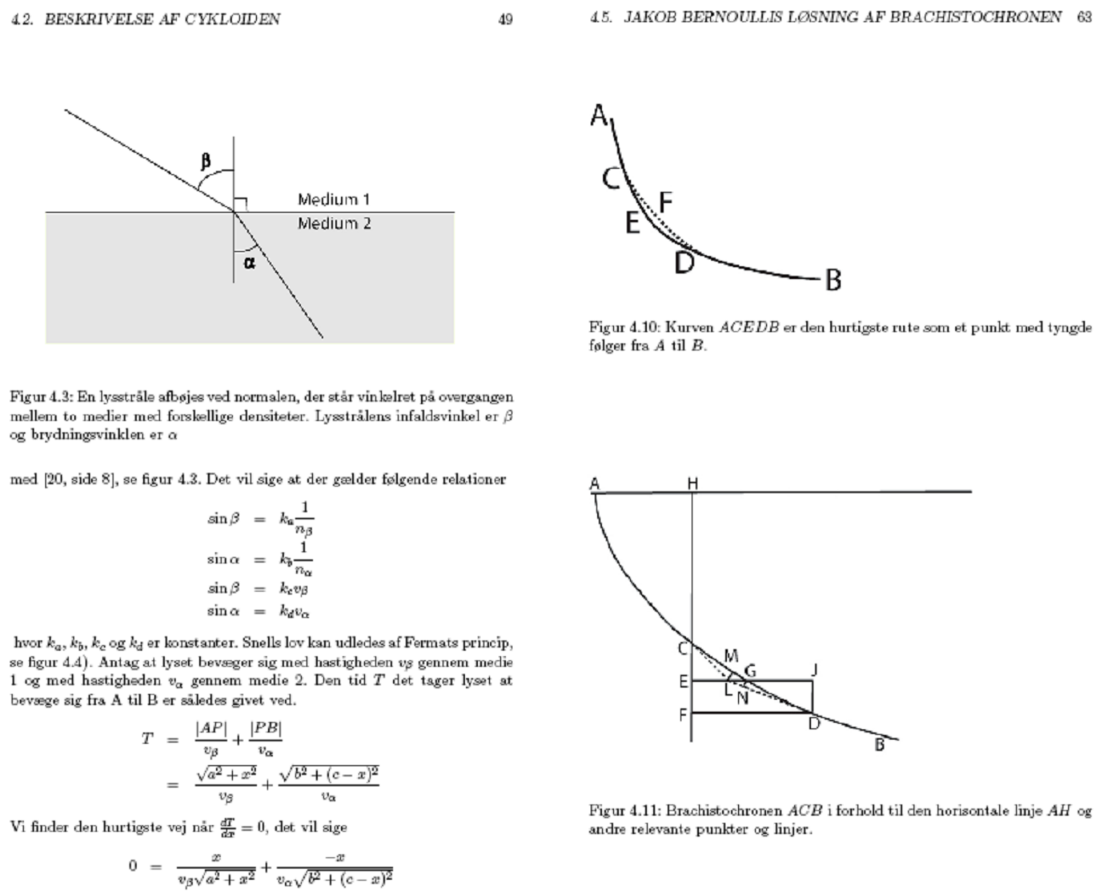
> equation of the catenary was paraphrased in that section. Whereas he later became much briefer in his derivations, he was very particular in this derivation. [$\cdots$]. We interpret this as if Johann Bernoulli felt a need to document that precisely this transformation from physics to differential equations was well founded. This was done with geometry which had a high degree of validity in this period." (Nielsen et al. 2005, 41)

> "Bernoulli did not know the logarithmic function, so he could not describe the curve [of the catenary] analytically. Even though he showed an incredible geometrical and mathematical intuition, his construction [of the solution] did not lead to a general solution of similar problems." (Nielsen et al. 2005, 42)

I will give one last example from the project work that illustrates how the students' mathematical competencies were trained. It is taken from the students' comparison of the Bernoulli brothers' different ways of solving the brachistochrone problem. Johann interpreted the moving point as a light particle that moves between two points. He used Fermat's principle of refraction and derived an equation involving the infinitesimals dx and dy. Jakob used a different strategy. He considered the problem as an extremum problem, using that an infinitesimal change in the curve would not increase time, due to the minimum property of the brachistochrone. Figure 4) shows two pages from the students' report, illustrating the two different approaches. The page to the left is from the students' treatment of Johann's solution and the page to the right is from their investigation of Jakob's solution.

In this part of their project work, the students' mathematical thinking competency was evoked and trained. The students experienced the characteristics of the nature of mathematics that makes it possible to generalize solution methods beyond particular, concrete problems. They wrote the following about the differences between Johann's and Jakob's approaches:

> "[$\cdots$] makes it [Johann's solution] weak from a mathematical point of view. His solution cannot be generalized because it was based on the physical situation. [$\cdots$] As indicated in section 4.5 Jakob's solution gave rise to the mathematical discipline of calculus of variations. His method



4.2. BESKRIVELSE AF CYKLOIDEN 49

Figur 4.3: En lysstråle afbøjes ved normalen, der står vinkelret på overgangen mellem to medier med forskellige densiteter. Lysstrålens infaldsvinkel er β og brydningsvinklen er α

med [20, side 8], se figur 4.3. Det vil sige at der gælder følgende relationer

$$\begin{aligned}\sin\beta &= k_a\frac{1}{n_\beta}\\ \sin\alpha &= k_b\frac{1}{n_\alpha}\\ \sin\beta &= k_c v_\beta\\ \sin\alpha &= k_d v_\alpha\end{aligned}$$

hvor k_a, k_b, k_c og k_d er konstanter. Snells lov kan udledes af Fermats princip, se figur 4.4). Antag at lyset bevæger sig med hastigheden v_β gennem medie 1 og med hastigheden v_α gennem medie 2. Den tid T det tager lyset at bevæge sig fra A til B er således givet ved.

$$\begin{aligned}T &= \frac{|AP|}{v_\beta} + \frac{|PB|}{v_\alpha}\\ &= \frac{\sqrt{a^2+x^2}}{v_\beta} + \frac{\sqrt{b^2+(c-x)^2}}{v_\alpha}\end{aligned}$$

Vi finder den hurtigste vej når $\frac{dT}{dx} = 0$, det vil sige

$$0 = \frac{x}{v_\beta\sqrt{a^2+x^2}} + \frac{-x}{v_\alpha\sqrt{b^2+(c-x)^2}}$$

4.5. JAKOB BERNOULLIS LØSNING AF BRACHISTOCHRONEN 63

Figur 4.10: Kurven $ACEDB$ er den hurtigste rute som et punkt med tyngde følger fra A til B.

Figur 4.11: Brachistochronen ACB i forhold til den horisontale linje AH og andre relevante punkter og linjer.

Figure 4: Page 49 and 63 of the students' report. Johann's (page 49) and Jakob's (page 63) approaches to the brachistochrone problem. The text is in Danish

> could be used to solve other such kinds of optimization problems. [· · ·] in contrast to his brother, Jakob was able to abstract from the mechanical framework of the problem of the brachistochrone. He separated the mathematics from the physics." (Nielsen et al. 2005, 75)

As the students explained, the technique Johann Bernoulli used to solve the problem was tied to the physical situation. It could not be generalized. Its scope was limited to the actual situation. In contrast, Jakob Bernoulli's solution method was independent of the concrete situation and could be used beyond that on other kinds of extremum problems—eventually leading to a new sub discipline in mathematics, called the calculus of variations.

As we have seen, this project work created complex teaching and learning situations where a scholarly approach to history was used that 1) developed the students' second order competency of having historical insights and possess historical awareness, 2) invoked and trained the students' mathematical competencies, 3) exhibited metalevel rules of mathematical discourse and made them objects of students' reflections. The students wrote their report using LaTeX and constructed the figures in their report using MatLab. If we include such skills in the competency to handle tools and aids of mathematics, the students were trained in all eight main competencies, as well as in the second order competency of developing historical awareness and insights into the history of mathematics.

The historical context provided situations where the students came to reflect upon differences between the sources' and their textbook's way of conceptualizing differential equations and their solutions, differences of argumentation, style and rigour. On the object level of mathematics discourse, this project work benefited the students' learning of mathematics especially through their discussions of why the Bernoulli brothers' use of infinitesimals as actual quantities gave "correct" answers despite the lack of rigour as we understand it today, where the concept of a function and of limit are crucial, concepts the Bernoulli brothers' did not have at their disposal. Through these discussions—dialogues—with the original sources, the students were forced into reflections upon their own understandings of the involved concepts on a structural level that went far beyond their initial operational dominated conception of differential equations.

5.2 Analysis of an experimental course in problem oriented project work: Egyptian mathematics.

The subject matter of the experimental course that will be analyzed in this section was Egyptian mathematics. The experimental course was developed in 2004 as part of an in-service course for upper secondary mathematics teachers in the Danish high school. The objective of the in-service course was to support mathematics teachers in developing, implementing and documenting problem oriented, group organised project work in history of mathematics in upper secondary mathematics education.[22] The in-service course began with a three day seminar during which the teachers in groups designed and developed a problem oriented project work of their own choice with specific learning objectives, course materials and products. Afterwards the teachers implemented their experimental course, i.e. their problem oriented project work, in one of their classrooms. During the experimental course, the teachers observed their students' group work in class. When the experimental course was finished, the teachers wrote a report documenting the implementation of their experimental course. The teachers' experiences with developing and implementing the problem oriented project work in history of mathematics in their classrooms where then discussed on the basis of these reports during a two-day seminar at the end of the in-service course.

The teacher, who developed and implemented the experimental course that will be analyzed in the following, chose Egyptian mathematics for several reasons. First of all, there is a textbook on Egyptian mathematics with sources translated into Danish.[23] And second, the course was meant to be interdisciplinary with history, and Egypt was suitable as a common theme that the mathematics teacher and the history teacher could agree upon.[24] The teacher's design of the experimental course was guided by his formulation of seven objectives for the students' learning. Four of these dealt with issues relating to independent study skills, and three of them concerned the history of mathematics as it was required in the new (2005) curriculum for mathematics in Danish high schools. In the following I will concentrate on his learning objectives regarding history and mathematics, which were the following: He wanted to

1. "have the students appreciate that mathematics has been different from what it is today

[22]History of mathematics and design and implementation of group and project organized teaching was part of a new curriculum in the Danish upper secondary school system (gymnasium) which was implemented in 2005.

[23]There exists only very little materials in Danish with sources from episodes in the history of mathematics. Lack of suitable resources is a major obstacle to introducing history for the learning and teaching of mathematics on a broader scope.

[24]Only the mathematics teacher participated in the in-service course.

2. develop the students' awareness that mathematical results have evolved, that mathematics is not static, which is contrary to the way it is often presented
3. develop the students' awareness that mathematics develops in an interplay with culture and society." (Wulff 2004, 2–3; my translation)

The teacher designed the project work in three phases: First he gave an introduction to Egyptian mathematics. He used two 45-minute lessons where he introduced the class to the Egyptian number symbols, showed them how the Egyptians multiplied numbers by repeating doubling and how they formulated problems. Second, the students were divided into groups of four. Each group worked with a chapter from the textbook on Egyptian mathematics: fractions, Pesu (bread and beer) exercises; first degree equations; two equations with two unknowns and second degree equations; the circle and approximations of π; the volume of a truncated pyramid; and computations of areas. Each group's work was guided by the problem formulation, chosen by the teacher: *How and why did the Egyptians calculate?* Eight lessons of 45 minutes were used in class for the independent group work (and an unknown amount of homework). During the group work, the teacher functioned as a consultant the students could call on for advice. Third, each group had to share the knowledge they had acquired in the group work. This was done in the form of a seminar where each group presented their work and their answers to the problem formulation supported by a power point presentation.

I will not go into further details about the work done in the groups,[25] but concentrate on using the theoretical framework to analyze the implementation of the experimental course and the teacher's evaluation. In his report, the teacher wrote about objectives 1) and 2) that "they were all about gaining insight into current mathematics precisely by studying the mathematics of another time" (Wulff 2004, 3). Hence, we are dealing with a use of the past from a utility perspective. In this part of the project work, the teacher took a pragmatic approach to history. This is also consistent with the teacher's attention in the classroom as he revealed in his report where he wrote: "Already during the first module [the first two lessons] came the classical question, why are we going to learn this? And we had a nice talk about the intended learning issues [1), 2) and 3) above], during which the class apparently accepted that historical mathematics, besides being interesting as such, could contribute to a more nuanced view on current mathematics." (Wulff 2004, 5). In the teacher's evaluation of to what degree the leaning objectives were realized, he thought that the students did not experience that mathematics develops over time, since the historical process of change was not dealt with in the project work. The students compared Egyptian and modern mathematics, through which they became aware that there were fundamental differences, i.e. the students experienced that mathematics has changed, but they did not study the actual process of change and how such processes come about. For the third learning objective, the teacher left the utility perspective and took a scholarly approach to history, as he wrote: "here is where the subject of history can be involved. From a general knowledge about Ancient Egypt and its society, students can discuss how society and culture have been driving forces for the mathematics of that time. At the same time the historians' method of source criticism is an essential tool for interpreting ambiguous and defective papyri" (Wulff 2004, 4).

The activities that guided the students' work, i.e. reading the sources and working with exercises presented in their respective chapters of the textbook on Egyptian mathematics can be interpreted as

[25]Interested readers are referred to Kjeldsen (2012) where some examples from the topics the students worked with are presented.

a kind of "living history" approach. The students had to learn and simulate how ancient Egyptians calculated, how they worked with geometry, posed mathematical problems etc. In doing so the students used different learning strategies and came to reflect upon mathematics on a structural level as can be seen from the following passage in the teacher's report: "Many students wondered about how "stupid" the Egyptians were. Why did they only use unit fractions? Why should a number be expressed as a sum of different unit fractions? On the other hand their methods were very difficult to understand; that is rather advanced, so in that respect they weren't stupid at all. I think that many of the students realized that current mathematics is not "just" like today, but is a result of a long development, during which many things have been simplified. [$\cdots$] This [that mathematics had made progress] became especially obvious when the students constantly rewrote the Egyptian notation to current notation with x's, formulas, etc. After they had finished an Egyptian calculation they would say: 'but that just corresponds to $\cdots$' followed by a solution of an equation in our way. It was very inspiring to see how students, who normally were a bit alienated towards x's and equations now had taken those to themselves as their own, and all of a sudden perceived equations as an easy way to solve problems. The students became aware that modern notation makes the calculations much easier than they would have been otherwise" (Wulff 2004, 7).

The teacher evaluated the experimental teaching course as partly successful. All learning goals except the last one (the third one above) were fulfilled. The last objective was suppose to have been reached through the students' work with answering the "why" part of the problem formulation – that is, why did the Egyptians calculate. The teacher had hoped that through interdisciplinary work with the students' school subject of history and their history teacher, the students would have experienced concrete examples of developments of mathematical ideas driven by needs of society. That didn't happen because the history teacher focused on other issues. The teacher reported that afterwards the students seemed to posses a more mature and reflective attitude towards mathematics.[26]

In this problem oriented project work, the teacher used past mathematics in different ways, from different perspectives and for different purposes. He had the students deal with history of mathematics from its practice, having the students work with the content matter of past mathematical text from the perspective of which techniques the Egyptians used and the kind of problems they worked with. For parts of the learning objectives the teacher used a pragmatic approach to history and for other parts he used a scholarly approach. The above quotes from the teacher's report show that several of the students' mathematical competencies were invoked and trained through this project work on Egyptian mathematics, especially their problem tackling skills and their competencies to deal with different representations of mathematical entities and to handle symbols and formalisms.

6 Discussion and concluding remarks

The theoretical framework presented in section 2), 3) and 4) draws on theories from mathematics education and from historiography adapted to history of mathematics and to history of mathematics in mathematics education. It is composed in such a way that it can deal with two central questions in research in integrating history of mathematics in mathematics education: 1) how integrating history

[26]The teacher finished the report approximately three months after his experimental teaching course had finished. Unfortunately, the teacher did not give examples of how and in what sense the students had a more mature and reflective attitude towards mathematics.

of mathematics benefits students' learning of mathematics and 2) how uses of historical elements to support students' learning of mathematics develop students' historical awareness. The framework is broad enough to accommodate the richness of the spectrum of implementations of history in mathematics education, and it is narrow enough to function as a tool for analyzing, criticizing and orienting designs and implementations of history for the teaching and learning of and about mathematics. The framework captures (some) of the multifaceted ways in which history can benefit students' learning of and about mathematics.

The part of the framework that concerns historiography and different forms of history provides a set of concepts that can be used to explore and identify how history is or can be understood. By linking that part of the framework with the second part that concerns thinking about purposes of mathematics education and mathematics learning it becomes possible to clarify and distinguish between different purposes for integrating history in mathematics education in relation to the two central issues presented above.

The framework was used to analyse the design and implementation of the experimental teaching course in Egyptian mathematics. The analysis revealed that the teacher used different approaches to history for different purposes targeted towards different learning goals – some directed towards mathematics and some towards history of mathematics. He created a complex and rich learning situation were the students developed new learning strategies, enhanced several of their mathematical competencies, and gained insights into the history of practices of mathematics. The analysis of the project work on physics' influence on the development of differential equations that was carried out by a group of five students in a university master's programme in mathematics showed that the students' used a scholarly approach to history for enlightening purposes. Their approach can be characterized as a multiple perspective approach to history of mathematics from its practice, adapted to mathematics education by focusing on the perspective of whether, how, for what purposes and to what degree the historical actors were inspired by problems from physics. The analysis revealed that in the process of exploring the three original sources, chosen by the students', on their own terms, the students identified, discussed and reflected upon differences between the historical actors' mathematical practices and the ones presented in their textbooks. Hereby, connections were created in the learning situation between the students' experiences with the involved mathematics from their textbook and their historical experience. These connections challenged the students to use other aspects of their mathematical conceptions in new situations provided by the historical context. The multifaceted ways in which history can benefit students' learning of mathematics became visible by employing the second part of the theoretical framework. It showed that the students were trained in all the main mathematical competencies, that they gained insight into history of mathematics in the sense of the second order competency of the KOM-project, and that they came to reflect upon meta-discursive rules in mathematics.

Finally, the combination of a multiple perspective approach to history of mathematics studied from practices of mathematics and Sfard's theory of thinking constitutes a foundation, from which it can be argued that history can function at the core of the learning of mathematics. Since meta-discursive rules in mathematics are contingent, they can be objects of historical investigations. The analysis of the students' project in history of differential equations showed that, by having original sources play the role as "interlocutors", differences in metalevel rules in the discourse of the sources, the students' textbooks, and/or their instructor and themselves, can be revealed. Hereby metalevel rules are exhibited and can be made the object of students' reflections. This indicates that history might be an obvious strategy for detecting students' meta-discusive rules and for students to develop proper metarules. The first part is explored in (Kjeldsen and Petersen, forthcoming) where it was possible to detect (some) improper meta-rules in students' mathematical discourse through a teaching module in history of the concept of a function that was implemented by the use of a matrix-organization that provided the teacher with a window into students' meta-rules. The second part, whether this caused a change in these students' metarules, is another question. To answer this question, more research is needed. However, knowledge about students' improper metarules can be used by a teacher to target further teaching goals in ways that focus students' attention towards developing proper meta-discursive rules.

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