MEANING OF THE METHOD OF EXCESS AND DEFICIT

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ABSTRACT

Young Bujok(盈不足, Ying Buzu in Chinese) is the title of the 7th chapter of The Nine Chapters on the Mathematical Art(九章算術), one of the oldest and the most influential ancient mathematical books. Young Bujok mathematically means the method of excess(盈) and deficit(不足). It is well-known that this method was transmitted from China to Europe via Arabia, becoming the rule of double false position.

We investigate the method of excess and deficit to conclude that it was a mathematical product of the creativity of the ancient Asians. It can be considered as a trial for construction of mathematical theories by Chinese, with the following chapters : Retangular Arrays and Right-angled Triangles. Even though the method has not been developed as a modern mathematical theory, we still possibly compare the method to the division algorithm and/or the modulo arithmetic via the concepts of division described by 實如法而一 and cross multiplication(維乘), from the point of view to deal with the remainders of divisions.

Keywords: Young Bujok(盈不足, Ying Buzu in Chinese, the excess and deficit), The Nine Chapters on the Mathematical Art(九章算術), the rule of double false position, cross multiplication(維乘)

1 Introduction

The Nine Chapters on the Mathematical Art(九章算術, Gu Jang San Sool in Korean, Jiu Zhang Suan Shu in Chinese) is one of the oldest and the most influential ancient mathematical books in East Asia, as like as Elements by Euclid in the West. Young Bujok(盈不足, Ying Buzu in Chinese) is the title of the 7th chapter of The Nine Chapters on the Mathematical Art (the Nine Chapters, in short). While we let Young Bujok itself to mean the chapter, it mathematically means the method of excess(盈) and deficit(不足). It is well-known that this method was transmitted from China to Europe via Arabia, becoming the rule of double false position [1, 5].

Chang Hyewon raised a question in [1] about the origin of such mathematical knowledges as the method of excess and deficit. She said that it was relatively easy to clarify the validity of the algorithms depicted in ancient mathematical books, but hard to figure out the origins of such mathematical knowledges, and only some conjectures about the origins were possible.

However, the method of excess and deficit looks very natural if we take for granted that the ancient Asians were clever and accustomed to division. The method, even though novel, ingenious in theory and powerful in applications, looks very alike to elementary number theory, for example, division algorithm.

With this thought, we here look for a mathematical and historical meaning and value of Young Bujok, the chapter of the method of excess and deficit. We first set up a viewpoint and/or an attitude for analysing Young Bujok. We then take a close look for the problems and the method of Young Bujok. We finally conclude that the method of excess and deficit is alike division algorithms, and it is very novel and ingenious mathematical product of the creativity of the ancient Asians.

2 Viewpoints and attitudes for the analysis on Young Bujok

We are going to adopt threefold viewpoints/attitudes for the analysis of the mathematical and historical meaning and value of the chapter Young Bujok.

We first adopt the attitude of Liu Hui, an ancient Chinese mathematician who made commentaries on the Nine Chapters. He himself told his attitude to look at the mathematics of the Nine Chapters in his preface of it [2, p. 56]:

> 徽幼習九章 長再詳覽 觀陰陽之割裂 總算術之根源 探賾之暇 遂悟其意.
> 是以敢竭頑魯, 采其所見, 爲之作注. 事類相推, 各有攸歸.
> When he was young, Lui Hui studied the Nine Chapters. Getting old, he closely investigate it again, and got to be aware of the change of Um Yang(陰陽, Yin Yang in Chinese) and the roots of arithmetic(mathematics). And researching the relationships(gaps) among them, he at last got the very meanings.
> Inspite of his dullness, he gathered his opinions and made commentaries. After his investigations and classification of things, (he concluded that) there are principles (of their own).

The above translation here is somewhat rough. However, since the ancient Chinese and English are a lot different form each other, it probably better to translate so as to figure out the meaning clearer. For the word by word or more detailed translation, refer to [5].

We adopt such attitude of Liu Hui so that we closely investigate each problem in Young Bujok to be aware of the root principles. So we minimize the references rather than widen, to analyse the mathematical and historical meaning and values of Young Bujok. We are rather going to only research on Young Bujok itself.

Secondly, we adopt Hawking's *model-dependent realism*, which can be clearly explained in the following quotation from *The Grand Design* [3] by S. Hawking and L. Mlodinow:

> Until the advent of modern physics it was generally thought that all knowledge of the world could be obtained through direct observation, . . . (omitted) . . . that is not the case. The naive view of reality therefore is not compatible with modern physics. To deal with such paradoxes we shall adopt an approach that we call *model-dependent realism*. . . . (ommitted) . . . But there may be different ways in which one could model the same physical situation, with each employing different fundamental elements and concepts. If two such physical theories or models accurately predict the same events, one cannot be said to be more real than the other; rather, we are free to use whichever model is most convenient.

In this quotation, we specially focus on the last statement: If there are several models which predict the same events, we are free to use the most convenient model. In this sense, we make a kind of model under which we understand the given situations, we mean here, the problems of Young Bujok.

Lastly, we look at the functions of mathematical problems. We know that there are problems simply for joy or time consumption, with no specific purposes, for example, some puzzles like cross-word puzzle and *sudoku*. However, problems in mathematical books are not like that. They have their own meaning and purposes. Such meaning and purposes are possibly roughly divided into two parts: one is the introduction/explanation of a theory, and the other is for practices/applications of the theory.

Since the Nine Chapters, and so the chapter Young Bujok also, consist of problems, we are going to think about the purposes of the problems of Young Bujok.

3 Problems and the method of Young Bujok

The Nine Chapters on Mathematical Art consist of nine chapters(books) with total 246 problems: chapter 1 方田 (Field Measurement) of 38 problems, chapter 2 粟米 (Millet and Rice) of 46 problems, chapter 3 衰分 (Distribution by Proportion) of 20 problems, chapter 4 少廣 (Short Width) of 24 problems, chapter 5 商功 (Construction Consultations) of 28 problems, chapter 6 均輸 (Fair Levies) of 28 problems, chapter 7 Young Bujok(盈不足, Excess and Deficit) of 20 problems, chapter 8 方程 (Rectangular Arrays) of 18 problems, and chapter 9 句股 (Right-angled Triangles) of 24 problems. For the details about the Nine Chapters, refer to [2], and for its English translation, refer to [5].

The chapter Young Bujok of the Nine Chapters consists of 20 problems as mentioned above. The first eight problems explain the method of excess and deficit, and the remaining twelve problems are the use of the the method in various kinds of situations, that is, the applications of the method of excess and deficit.

We closely look at the first eight problems, but are not going to discuss about the remaining twelve problems for now. We might have a chance to discuss about them later. The first 8 problems are:

1. 今有 共買物. 人出八, 盈三. 人出七, 不足四. 問人數, 物價各幾何?
   Here are something to buy together. If each person pays 8, then the amount exceeds the price by 3. If pay 7, then the deficit is 4 to the price. How many people are there? How much is the price?
2. 今有 共買鷄. 人出九, 盈十一. 人出六, 不足十六. 問人數, 鷄價各幾何?
3. 今有 共買璡. 人出半, 盈四. 人出少半, 不足三. 問人數, 璡價各幾何?
   To buy a jade together, if each pays a half, then the excess is 4, and if each pays a third, then the deficit is 3 to the price of the jade. How many people are there? How much is the price?
4. 今有 共買牛. 七家共出一百九十, 不足三百三十. 九家共出二百七十, 盈三十. 問家數, 牛價各幾何?
5. 今有 共買金. 人出四百, 盈三千四百. 人出三百, 盈一百. 問人數, 金價各幾何?
6. 今有 共買羊. 人出五, 不足四十五. 人出七, 不足三. 問人數, 羊價各幾何?
7. 今有 共買豕. 人出一百, 盈一百. 人出九十, 適足. 問人數, 豕價各幾何?
8. 今有 共買犬. 人出五, 不足九十. 人出五十, 適足. 問人數, 犬價各幾何?

We translate only two problems. But it is enough to see that these statements are in a certain pattern, and what problems mean. The problems are to say that: There is something for some people to buy together(今有 共買物). If each person pays(出) some money, then there be a remainder, an excess(盈) or a deficit(不足) to the price of that *something*. In case of problem 7 and 8, there is no remainder in one condition. Given two conditions, the number of people and the price are questioned.

After problem 4, the general method of excess and deficit is explained as

> 置所出率, 盈, 不足各居其下. 令維乘所出率, 并以為實. 并盈, 不足為法.
> 實如法而一. 有分者, 通之. 盈不足相與同 其買物者.
> 置所出率. 以少減多, 餘, 以約法, 實. 實為物價, 法為人數.[1]
> Put each of payments and each remainder(excess or deficit) under each of them. Make a cross multiplication(維乘) among the payments and the remainders. Let the sum of the multiplied payments be sil(實), and the sum of excess and deficit bub(法).
> Divide sil by bub(實如法而一). If there are fractions, reduce them to a common denominator.
> If the remainders are sumed up to zero, then it means the purchase together.
> Make a difference(餘) of the payments. Divide bub(法) by the difference(餘) to find the number of people. Divide sil(實) by the difference(餘) to get the price.

There are more explanations on problems 5 and 6, and 7 and 8, respectively. But these method are special cases of the above so that we omit them.

4 Analysis of Academic/Historical Meanning and Value of Young Bujok

We first think of the phrase 實如法而一 which comes out of everywhere in the Nine Chapters. Koh and Ree, in [4], looked for logic or mathematical formalism underneath the ancient Asian mathematics. They focused on sil(實) and bub(法). Sil(實, shi in Chinese) means a reality, that is, what one has in real, meanwhile bub(法, fa in Chinese) means a basic standard and/or a unit. So the phrase 實如法而一 is understood as that if what a person has is the same as the unit, then it becomes one. This, therefore, means the division, even though it does not mention about the remainder.

As we know, the determination of a unit motivates the inventions of the natural numbers, and then naturally additions of them. With some cleverness and creativity that human beings have, the multiplication naturally comes out. So, in some sense, division is a starting point of arithmetic or mathematics. Therefore, setting up 實 and 法, or the phrase 實如法而一, is considered as the very beginning of the ancient Asian mathematics.

We now look at the chapter Young Bujok. It starts with the explanation of its title [2, p. 119] as:

> 盈不足, 以御隱雜互見.

This means that the method of excess and deficit deals with complex and complicated situations to clear up and simplify them. This is a sort of fundamental and basic meaning and value of mathematics. In mathematics, fundamentally, we simplify complicated situations using given clues or conditions to figure out the situations(the present). With such anaysis of the situations, we can predict our near

[1] In [2] the statements of the method is a little bit differently given: 盈不足相與同 共買物者. 置所出率, 盈, 不足各居其下. 令維乘所出率, 并以為實. 并盈, 不足為法. 有分者, 通之. 副置所出率. 以少減多, 餘, 以約法, 實. 實為物價, 法為人數.

future, or invert the situation to find the causes(our near past). This is a value of doing mathematics. Young Bujok affords such privilege.

It is also important to notice that there is a mathematical pattern in the statements of the first eight problems in the chapter Young Bujok. For example, let a denote the number of people, and b the price of the thing to buy together in the problems. In problem 1, the first condition tells

$$8a = b + 3 \quad \text{or} \quad b = 8a - 3,$$

and the second condition tells

$$7a = b - 4 \quad \text{or} \quad b = 7a + 4.$$

Similarly, in problem 3, the the first and second conditions tell

$$\frac{1}{2}a = b + 4 \quad \text{or} \quad b = \frac{1}{2}a - 4,$$

$$\frac{1}{3}a = b - 3 \quad \text{or} \quad b = \frac{1}{3}a + 3.$$

These conditions look like the division algorithm: Given natural numbers $a \leq b$, there exist natural numbers q and r such that

$$b = aq + r \quad \text{with } 0 \leq r < a.$$

Combining 實如法而一 and the pattern of the given statements in Young Bujok, we see a similarity between the representation of problems in Young Bujok and division algorithm.

It is sure that there are big differences between them. For example, in division algorithm, only natural numbers are treated and the remainder r should be in the range of $0 \leq r < a$, a the divisor. Meanwhile, in Young Bujok, the dividend or 實 b and the divisor or 法 a, even the quotient q and the remainder r are not necessarily the natural numbers. They can be rational numbers, and even the remainder(excess or deficit) can be any number. However, using 實如法而一, we can divide 實 b by 法 a as much as we want, and then we get 盈 or 不足, which are considered as the remainder. In this way, Young Bujok combined with 實如法而一 is considered as a sort of generalized form of division algorithm.

There is one more thing worth noticing. Division algorithm is a sort of deductive computation, which means the quotient and the remainder are uniquely determined from the given dividend and divisor. However, Young Bujok problems are given in the form of inversion. We are given the quotient and the remainder, and asked to find the dividend and the divisor. In this sense, Young Bujok is much more *mathematical* rather than *arithmetical*.[2]

We then consider the meaning of the cross multiplication(維乘) as the management tool of the remainders(excess or deficit). For example, in problem 1, when a denotes the number of people and b the price, if each person pays 8, then there is an excess 3 of the price so that $8a = b + 3$. And if each person pays 7, then there is an deficit 4 of the price so that $7a = b - 4$. Now if we make a cross multiplication which means the multiplication of 4 and 3 to the former and the latter equations, respectively, to get

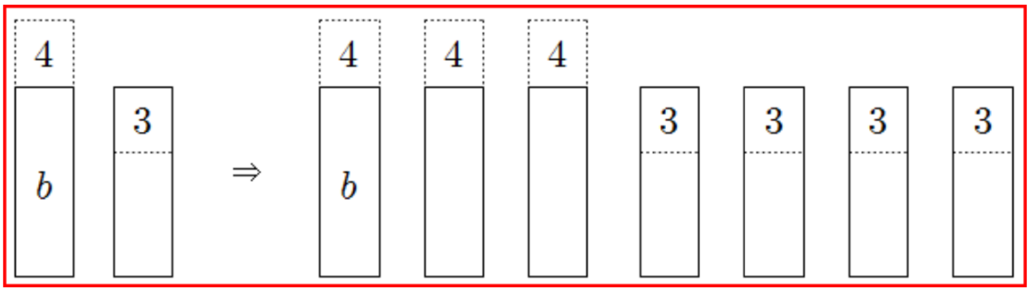
$$32a = 4b + 12 \qquad \text{and} \qquad 21a = 3b - 12,$$

[2]Needham insisted that the West owes to Chinese mathematics for the use of indeterminates. - in Martzloff, J. C., 1997, *A History of Chinese Mathematics*, Springer, Berlin.

and then add the equations up to get

$$(32 + 21)a = (4 + 3)b,$$

in which there is no remainder, that means, the cross multiplication is a method to find a multiple of a certain number, in this case, the price of the thing to buy together. In this sense, the cross multiplication is compared to modulo arithmetic in number theory. In the explanation of the method of excess and deficit, it is said 盈不足相與同 其買物者, which means that to give and take 盈 and 不足 to get rid of each other is just the purchase together of the thing (in multiple). This is depicted in the figure 1.



⟨ Fig. 1 ⟩ cross multiplication

The figure shows that the utility of cross multiplication. Even though we want to buy a thing, we duplicate the situation of common purchase to get rid of remainders(excess or deficit). The number of duplications are given by the excess and the deficit. This is the ingeneous, even though simple, creative mathematical product of the ancient Asians.

We finally consider the nature of the problems in Young Bujok. As mentioned above, the first eight problems have the purpose of the introduction and explanation of the mathematical theory, the method of excess and deficit. In fact, the first four problems provide a prototype and the other four give some variations. But if we write these variations in modern mathematical terminology, then it is easy to see that they are surely contained in the prototype.

Even though we haven't touched on the application problems which come after the first eight problems, the application is easily derived from the the method of excess and deficit, which are now considered as the rule of double false position.

If we look at the title of the Nine Chapters, then we can feel a kind of practicality from the first six titles. It seems that the mathematics in the first six chapters would rather be used in governmental offices, for example, the office of taxes. But the latter three chapters, Young Bujok(Excess and Deficit), Retangular Arrays and Right-angled Triangles seem rather mathematical(theoretical). In fact, in the sense of providing a theoretical prototype in the first few problems, they are considered as an effort or a trial to make theories by the ancient Chinese mathematicians.

5 Conclusion

We investigate the method of excess and deficit to conclude that it was a mathematical product of the creativity of the ancient Asians. It can be considered as a trial for construction of mathematical theories by Chinese, with the following chapters : Retangular Arrays and Right-angled Triangles. Even though the method has not been developed as a modern mathematical theory, we still possibly compare the method to the division algorithm and/or the modulo arithmetic via the concepts of division described by 實如法而一 and cross multiplication(維乘), from the point of view to deal with the remainders of divisions.

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