EDUCATIONAL MEANING OF THE THEORY OF RECTANGULAR ARRAY IN NINE CHAPTERS ON THE MATHEMATICAL ART

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ABSTRACT

We first propose a principle of comprehensive cognitive development process by reviewing and synthesizing Piaget's theory of cognitive development, learning theory such as Dubinsky's APOS theory, and the epistemology of ancient Chinese philosopher Xun Zi(荀子). Our principle of cognitive ability development process comprises 4 stages: adaptation stage(familiarization of external stimulation); receptiveness stage(acceptance of information); completion stage(completion of conceptualization); utilization stage(application of knowledge). We think that a person experiences these 4 stages of cognitive process internally when he or she gets to acquire some knowledge.

We investigate the 8th chapter ⟨Rectangular Arrays(方程)⟩ of the ancient Chinese mathematics book ⟨Nine Chapters⟩[1] on the Mathematical Art(九章算術)⟩. ⟨Rectangular Arrays(方程)⟩ deals with the theory of systems of linear equations. We discuss an educational meaning and the value of ⟨Rectangular Arrays⟩ from the viewpoint of our principle of cognitive ability development.

1 Introduction

Euclid's ⟨Elements⟩ is regarded as the origin of systematically organized western mathematics. Around the time when ⟨Elements⟩ was written, an anonymous ancient Chinese mathematical book, ⟨Nine Chapters on the Mathematical Art(九章算術)⟩ was compiled during the Former Han(前漢) Dynasty (206 BC–8 AD). Some contents of the book date back to before Qin(秦) Dynasty (221–207 BC)[3]. Since the book had been compiled, several annotations had been added to the book, among which the commentary made by the first annotator Liu Hui(劉徽) in AD 264 is highly evaluated in that he supplemented the book using his own approach based on novel theories and ideas.

⟨Nine Chapters⟩ is considered one of the oldest and the most influential ancient Asian mathematical books. It had constituted the basis of eastern mathematics until the Western mathematics was introduced. The book is comprised of 246 practical problems and solutions which are categorized into nine chapters, namely, 1. rectangular fields(方田), 2. millet and rice(粟米), 3. distribution by proportion(衰分), 4. short width(少廣), 5. construction consultation(商功), 6. fair levies(均輸), 7. excess and deficit(盈不足), 8. rectangular arrays(方程), 9. right triangles(句股).

[1] ⟨Nine Chapters on the Mathematical Art⟩ is shortened to ⟨Nine Chapters⟩.

⟨Nine chapters⟩ is well organized in the pedagogical point of view. The problems in each chapter in the book are arranged from easy to more complicated ones to solve. We here are particularly interested in chapter 8 ⟨Rectangular Arrays(方程)⟩. It deals with a system of linear equations and the solution method given there is essentially the same as the Gaussian elimination method for a matrix.

In this paper, we investigate the structure of the chapter 8 in terms of the learning process or cognitive development process of human mind. To do this, we first propose a principle of comprehensive cognitive developmental process by reviewing and synthesizing Piaget's theory of cognitive development, learning theory such as Dubinsky's APOS theory, and the epistemology of ancient Chinese philosopher Xun Zi(荀子).

2 Cognitive ability development process

Human's cognitive ability is cultivated and developed from education. Education is the process which provides human with resources and experiences so that he or she may acquire knowledge and various skills to carry out any task while living as a human. In that sense education is very important and indispensible for human life. The more knowledge and information one has, the faster one's cognitive ability expands. The further one's cognitive ability grows, the faster one absorbs knowledge and skills. It is not hard to believe that education is very closely related to human cognition ability and so education should be designed in accordance with human cognition level. So when a book is written for the purpose of teaching, reader's cognitive ability development process should be considered.

In this section, we propose a principle of cognitive ability development, which has much in common with each of Piaget's cognitive developmental theory, Dubinsky's mathematics learning theory and the epistemology of Xun Zi(荀子).

2.1 Piaget's theory of cognitive development

Piaget ascertained some patterns and strategies of thought in each of the four cognitive developmental stages: sensorimotor stage, preoperational stage, concrete operational stage, formal operational stage. These stages are divided according to children's age.

He also suggested a developmental process that is consisted of a cycle: The child performs an action which has an effect on or organizes objects, and the child is able to note the characteristics of the action and its effects(*action*); Through repeated actions, perhaps with variations or in different contexts or on different kinds of objects, the child is able to differentiate and integrate its elements and effects(*process of reflecting abstraction*) and the child is able to identify the properties of objects by the way different kinds of action affect them(*process of empirical abstraction*); By repeating this process across a wide range of objects and actions, the child establishes a new level of knowledge and insight(*process of forming a new cognitive stage*); The child use them to create more complex objects and to carry out more complex actions(*creation*); As a result, the child recognize still more complex patterns and construct still more complex objects and new stage begins [5].

Piaget's cognitive development theory forms the foundation of the constructivism, which is a psychological theory on cognition. It says that knowledge and meaning are constructed by individual experiences and so learning and thinking require participation of the learner. Piaget believed that knowledge is constructed and re-constructed by the learner by interacting with objects [7].

2.2 Dubinsky's APOS theory

Dubinsky made a constructivist theory of learning in undergraduate mathematics by extending to college level the reflective abstraction in children's learning in Piaget's developmental process. His theory is based on the idea that an individual acquires mathematical knowledge by constructing mental actions, processes, and objects and organize them in schemas to make sense of situations and solve problems.

An *action* is a transformation of individual's externally perceived object by step by step instructions on how to perform the operation. When an action is repeated and the individual reflects on it, he or she make an internal mental construction called a *process*. When the individual becomes aware of the process as a totality and realize that transformations can act on it, an *object* is constructed. A *schema* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked to form a framework in the individual's mind [1].

2.3 The epistemology of Xun Zi(荀子)

While the logical account for the cognitive process started to be given from Piaget's work in the west, there were attempts to make a logical interpretation of the human cognitive process in the east way long before Piaget.

A Chinese Confucian philosopher Xun Zi(荀子, 312–230 BC) pointed out that human mind is working through 4 stages: nature of things(性), human feeling of things(情), human thinking of things(慮), human action(僞) [2]

That is to say, when one perceives a thing, he or she has a feeling about it and then he or she think about it and then finally takes an action according to the thought. This means that after obtained by sensory perception, knowledge or information is organized and readjusted in accordance with one's recognition, and then it becomes utilized.

2.4 A principle of cognitive ability development

Education is the process through which knowledges are accumulated and cognitive ability is cultivated. Education process helps each individual build a thinking system. From well established thinking system comes the ability to concretely operate concepts according to each individual's cognitive level. By enhancing the cognitive ability, intellectual capacity is expanded and a series of actions leading to the development of intellectual capacity is understood as a main factor of education.

We propose that the cognitive ability development process in individual's mind is comprised of 4 stages, namely, *adaptation stage*(familiarization of external stimulus), *receptiveness stage*(acceptance of information), *completion stage*(completion of conceptualization), and *utilization stage*(application of knowledge).

In the adaptation stage, an individual shows rather passive attitudes to the external stimuli such as new concepts, new information, or new methods. He or she just senses the stimuli, not showing much interest. When one is getting familiarized with the repeating stimuli and gets interested in them, he or she moves on to the next stage.

In the receptiveness stage, one takes more active part to accept the new concepts and methods. He or she takes time and efforts to reflect and practice in order to accept them. As the external stimuli

are maturing internally by one's reflections and practices, he or she gets to be more and more appreciating and understanding them. This concept-maturing-process is indispensable for knowledge establishment.

In the completion stage, a concept is acquired and fully understood so it becomes established knowledge in one's mind, ready to be utilized whenever needed. In the utilization stage, one can apply the established knowledge in order to interpret various phenomena and to solve real world problems. In this stage, intellectual capacity is further expanded and the cognitive ability is more promoted than before.

One notable feature of the principle of cognitive ability development is what distinguishes adaptation stage from receptiveness stage. In the adaptation stage the cognitive level allows no more than rote or mechanical learning, while in the receptiveness stage self-directed learning based on constructivism takes place. Individual's action and experience play very important roles in concept maturing which brings knowledge establishment.

Our principle of cognitive ability development has much in common with each of Piaget's cognitive developmental process, Dubinsky's APOS theory, and Xun Zi's epistemology. Following table shows how these theories can be compared.

	adaptation	receptiveness	completion	utilization
Piaget	action	reflective & empirical abstraction	forming a new cognitive stage	creation
Dubinsky	action	process / object	schema	
Xun Zi	nature	feeling	thinking	thoughtful action

3 Rectangular Arrays(方程) and the cognitive ability development process

The 8th chapter ⟨Rectangular Arrays⟩ of ⟨Nine Chapters⟩ deals with systems of linear equations. The solving method used here is ⟨rectangular array method(方程術)⟩, which is essentially the same as the Gaussian elimination method. The Gaussian elimination method was named after Carl Friedrich Gauss, but it was not invented by him. The method stems from the notes of Isaac Newton in 17th century [4]. This fact indicates that the theory of a system of linear equations was developed in the east much earlier than it appeared in the west.

In this chapter, a rule for calculating negative numbers, called the ⟨sign rule(正負術)⟩, is also introduced. Shen mentioned that it is surprising to find that Liu Hui summed up the ⟨sign rule⟩ 1700 years ago in the same way we understand and apply now. Such a treatment noted in the west is not found until after the Renaissance [3].

We briefly discuss the characteristics of each problem in light of our cognitive ability development process below. We present the original text and the corresponding English translation of some of the problems to help readers grasp what the chapter is all about.

第一問. 今有 上禾三秉 中禾二秉 下禾一秉 實三十九斗. 上禾二秉 中禾三秉 下禾一秉 實三十四斗. 上禾一秉 中禾二秉 下禾三秉 實二十六斗. 問上中下禾一秉各幾何?

Problem 1. Now there are 3 sheaves of top grade rice, 2 sheaves of middle grade rice, 1 sheaf of

low grade rice. Total volume is 39 dou[2]of rice; 2 sheaves of top grade rice, 3 sheaves of middle grade rice, 1 sheaf of low grade rice. Total is 34 dou; 1 sheaf of top grade rice, 2 sheaves of middle grade rice, 3 sheaves of low grade rice. Total is 26 dou. What is the volume of 1 sheaf of each grade rice?

The ⟨rectangular arrays(方程)⟩ and the ⟨rectangular array method(方程術)⟩ are introduced as a way of solving problem 1, which is represented by a system of linear equations with 3 unknowns in modern algebra. The solving method suggested in ⟨Nine Chapters⟩ can be briefly explained as follows: First of all, write down the given numbers of things in each column, making a ⟨rectangular array⟩, or a 4×3 matrix. Then eliminate small top entries by performing a series of column operations until the resulting array becomes a lower triangular matrix, from which final answers can be obtained by division and substitution. This method is called the ⟨rectangular array method⟩, which is the same as the Gaussian elimination method. Below shown are the starting array and the ending array for this problem.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{pmatrix}$$

第 二 問. 今有 上禾七秉 損實一斗 益之下禾二秉 而實一十斗. 下禾八秉 益實一斗與上禾二秉 而實一十斗. 問上下禾實一秉各幾何?

Problem 2. Now there are 7 sheaves of top grade rice, subtracting 1 dou from them and adding 2 sheaves of low grade rice makes 10 dou; 8 sheaves of low grade rice, adding 1 dou and adding 2 sheaves of top grade rice makes 10 dou. What is the volume of 1 sheaf of each grade rice?

Problem 2 can be expressed using modern notation by linear equations with 2 unknowns

$$7x - 1 + 2y = 10 \text{ and } 8y + 1 + 2x = 10.$$

Problem 2 explains transposing constant terms with the passage that "subtracting is adding (to another)[3]and adding is subtracting (to another) ". So the columns of the array are $(7, 2, 11)$ and $(2, 8, 9)$.

第 三 問. 今有 上禾二秉 中禾三秉 下禾四秉 實皆不滿斗. 上取中 中取下 下取上 各一秉 而實滿斗. 問上中下禾實一秉各幾何?

Problem 3. Now there are 2 sheaves of top grade rice, 3 sheaves of middle grade rice, 4 sheaves of low grade rice, the volume of each is less than 1 dou. Adding (1 sheaf of) middle grade rice to the top grade rice, (1 sheaf of) low grade rice to the middle grade rice, (1 sheaf of) top grade rice to the low grade rice makes 1 dou each. What is the volume of 1 sheaf of rice of each grade?

Problem 3 is to solve the system

$$2x < 1, \quad 3y < 1, \quad 4z < 1, \quad 2x + y = 1, \quad 3y + z = 1, \quad 4z + x = 1.$$

[2]Dou(斗) is a unit of volume.
[3]"損之曰益 益之曰損"

Here, negative numbers inevitably appear in the matrix reducing process. A method of addition and subtraction of negative numbers is explained, which is called the ⟨sign rule(正負術)⟩, where ⟨正⟩ and ⟨負⟩ refer to positive and negative, respectively. The ⟨sign rule⟩ is summarized as saying that "like signs subtract, opposite signs add (when subtracting); opposite signs subtract, like signs add (when adding)."

Problems 1, 2, and 3 provide a certain kind of problem situations, for which a new mathematics is required. These problems introduce a new concept such as ⟨rectangular array⟩, a new method such as ⟨rectangular array methods⟩, and a new rule such as ⟨sign rule⟩.

Exposing learners to such problem situations and introducing new concepts and new methods related with the problem is like providing a stimulus. This is the adaptation stage for learners. Providing a stimulus is in turn like sowing a seed. Like a big tree starts from a seed, knowledge is formed from a stimulus. A seed can hardly be seen at first but soon with a good care it will grow to show a little leaf and then it will be noticed. When a stimulus is repeatedly given to learners, who may not have recognized it at first, they will sense it stronger and stronger and adapt to it without deep understanding of it. As learners try intentionally to understand and get used to it, they are gradually accepting such stimulus as meaningful information or knowledge.

> 第四問. 今有 上禾五秉 損實一斗一升 當下禾七秉. 上禾七秉 損實二斗五升 當下禾五秉.
> 問上下禾實一秉各幾何?
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> Problem 4. Now there are 5 sheaves of top grade rice. If 1 dou 1 sheng[4] is subtracted from them, the rest is equivalent to 7 sheaves of low grade rice; 7 sheaves of top grade rice subtracted 2 dou 5 sheng is equivalent to 5 sheaves of low grade rice. What is the volume of 1 sheaf of rice of each grade?

In modern notation, the system $5x-11=7y, 7x-25=5y$ represents the problem 4. The suggested solution is to make an array consisting of two columns $(5,-7,11)$ and $(7,-5,25)$. We notice here that y terms are transposed as well as the constant terms and negative numbers appear as some entries of the array.

Problems 5 and 6, like problem 4, deal with linear equations with 2 unknowns involving negative coefficients, and they are solved by applying the concepts and rules introduced before. All the problems 1 through 6 are asking what the volume of 1 sheaf of rice of each grade is.

Problems 4, 5, and 6 intend to help learners get more familiarized to the problem situations and solving methods newly introduced before. These problems are simple variations of the problems in the adaptation stage. With these problems learners can practice applying new concepts and methods. While learners are taking time and efforts to figure out what the new stimulus or information means and to reflect on it, they become more receptive and experience the newly adapted concepts and methods maturing in their minds. This is the receptiveness stage of the cognitive ability development process. This is the period for which a very little plant is taken very good care of to grow continuously.

> 第七問. 今有 牛五 羊二 直金十兩. 牛二 羊五 直今八兩. 問牛羊直金其何?
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> Problem 7. Now there are 5 cows and 2 sheep, the total price is 10; 2 cows and 5 sheep, the total price is 8. What is the price of a cow and a sheep each?

[4]Sheng(升) is a unit of volume. 1 dou = 10 sheng.Sheng(升) is a unit of volume. 1 dou = 10 sheng.

Problem 7 deals with the price of domestic animals. According to the suggested solution given in ⟨Nine Chapters⟩, the resulting matrix is different from the one given in problem 1, where the first (from right) column remains the same throughout the whole process, while in problem 7 the first column (from right) has been multiplied as shown below. When a varied column operation is introduced, learners' computation skills will be improved.

$$\begin{pmatrix} 2 & 5 \\ 5 & 2 \\ 8 & 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 10 & 10 \\ 25 & 4 \\ 40 & 20 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 10 \\ 21 & 4 \\ 20 & 20 \end{pmatrix}$$

第 八 問. 今有 賣牛二 羊五 以買十三豕, 有餘錢一千. 賣牛三 豕三 以買九羊 錢適足.
賣羊六 豕八 以買五牛 錢不足六百. 問牛 羊 豕價各幾何?

Problem 8. Now sell 2 cows, 5 sheep, and buy 13 pigs, then there left 1000 coins; sell 3 cows, 3 pigs, and buy 9 sheep, then no coin is left nor deficient; sell 6 sheep, 8 pigs, and buy 5 cows, then there is 600 coins shortage. What is the price of a cow, a sheep, and a pig each?

Problem 8 involves the situation with zero constant term and negative coefficients as well. It has 3 unknowns and asks about the price of domestic animals. Problem 9 is an application problem with 2 unknowns asking the weight of birds.

第 十 問. 今有 甲乙二人持錢不知其數. 甲得乙半而錢五十. 乙得甲太半而亦錢五十.
問甲乙持錢各幾何?

Problem 10. Now there are two persons A and B, each of whom has unknown amount of coins. If A gets half of what B has then A will be having 50 coins. If B gets two thirds of what A has, then B will also be having 50 coins. What is the amount of coins A and B each has?

Problem 10, asking the amount of coins with 2 unknowns, shows how to deal with arrays including fractions. It suggests to multiply an appropriate number to the column including fractions to get integer entries. Problem 11, together with problem 10, extends the theory to more general situations to handle fractions. It asks about the price of domestic animals with 2 unknowns.

Problems from 7 to 11 require mixed calculations involving negative integers, fractions, and 0. They look the type of problems that have the intention to improve learners' calculation skills. While problems from 1 to 6 ask only about the volume of 1 sheaf of rice, various objects such as the price of domestic animals, the weight of birds, the amount of money are dealt with in these problem situations. These problems provide more complicated situations than the previous ones so the learners can establish knowledge completely which has been maturing all along. By facing a variety of problem situations and solving them on their own, learners get confident with the new concepts and methods, and new knowledge becomes fully established. Their cognitive levels are upgraded in this completion stage. The seed has completely grown to be a big tree ready to produce fruits.

Problem 12 deals with horse power with 3 unknowns. Problem 13 asks the length of ropes in the situation expressed with 5 equations and 6 unknowns. Problem 14 asks about the yield of each field with 4 unknowns. Problem 15 asks the weight of rice with 3 unknowns and negative coefficients. Problem 16 asks the number of chickens eaten up with 3 unknowns. Problem 17 asks the prices of domestic animals with 4 unknowns. Problem 18 asks the prices of grains with 5 unknowns.

The problem situations given in problems from 12 to 18 are still more complicated than before: the numbers of unknowns are getting larger; an indeterminate system is included; a variety of objects including abstract ones such as horse power should be dealt with. These various real world problems can be solved with careful application of established knowledge. Learners who have fully established knowledge can utilize it to solve application problems. So their cognitive ability is enhanced. This happens in the utilization stage. We can enjoy the fruits of the big tree which will produce many more fruits over the years.

4 The educational meaning of the ⟨Rectangular Arrays⟩

As we saw in the above section, the 18 problems in the chapter ⟨Rectangular Arrays⟩ are arranged from easy to complicated ones with respect to problem situations and solving method. More specifically, mathematical levels are upgraded from problems involving only positive numbers to ones with integers or fractions; the concepts included in problem situations extend from concrete ones such as volume and length to abstract ones such as horse power and price; the number of unknowns increases from 2 to 6 so the matrix operations are getting more complicated. One of the problems deals with an indeterminate system of linear equations.

In this section, we summarize what we analyzed the ⟨Rectangular Arrays⟩ chapter of ⟨Nine Chapters⟩ from the standpoint of cognitive ability development.

Problems 1, 2, and 3 are designed for introducing a certain kind of problem situations and solving methods so that learners can familiarize and adapt themselves to them. These are the problems for the adaptation stage.

Learners can practice solving problems by applying the rules instructed in the previous problems to similar situations given in problems 4, 5 and 6. These problems are meant to help learners become receptive to new concepts and methods and understand them. By facing and solving various problem situations repeatedly, learners can make new concepts and methods mature internally. This happens in the receptiveness stage and these problems are for the receptiveness stage.

Learners get the chance to face varied problem situations involving different objects starting from problem 7. By extending the methods to even more complicated problem situations as exemplified in problems from 8 to 11, learners can deepen their understanding and get confident with the newly developed concept. So they finally can reach to the stage of completion of knowledge.

Learners can apply their established knowledge on the subject in order to solve more complicated and practical real world problems such as problems from 12 to 18, and especially, to solve an indeterminate system of linear equations like problem 13. So these problems are for the utilization stage.

There are some other notable features of this chapter in educational viewpoint. The problems used to explain the methods ⟨rectangular array method(方程術)⟩ and ⟨sign rule(正負術)⟩ have 3 unknowns, whereas 2 unknowns are used when a rather sophisticated computation skill is needed. Application problems have mainly 3 or 4 unknowns with one problem with 5 and 6 unknowns each. These features coincide with our thoughts. When we teach a system of linear equations or matrices to students, we typically introduce the topic with problems with 3 unknowns and we say that the same method applies to more general situations involving more than 3 unknowns. If we have to deal with more complicated situations where a special computation technique is needed, then it will be best to start

with 2 unknowns and then generalize. In that sense, ancient Chinese mathematical book seems to reflect our way of thinking in terms of education.

5 Final Remark

⟨Nine Chapters⟩ was published way long before there appeared a theory of human cognitive development. Even so, the problems in each chapter are organized and arranged as if human cognitive development process is considered. The fact that psychology in the west was developed and established in 20th century does not mean that there had been no recognition of human mind or psychology before. When there was no theory on learning, cognition, or psychology, the compilers of ⟨Nine Chapters⟩ seemed to understand how human cognitive ability was developing and how to teach people efficiently.

Finally, the comment made by Liu Hui below problem 18 is worth noticing. Problem 18 is a rather complicated one involving 5 unknowns and in the commentary, Liu Hui encouraged readers to solve problems not by mechanically mimicking the solutions of previous problems but by applying a better and simpler method developed from one's own reflection and thought. Liu Hui meant that mathematical technique is developed by practicing simple operations and then combining them in more complicated ways to get to the solution of more difficult problems. And this is what we mean by completion of knowledge and utilization of it. His comment indicates that he understood mathematics as a subject for cultivating creative thinking, which would be what mathematics education intends to achieve now.

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