YANG HUI'S *NAYIN QILI*
楊輝의 納音 起例

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ABSTRACT

The sexagesimal cycle(干支) has been playing a very important role in ordinary human affairs including astrology and almanacs as well as the arts of divination(術數) since the second millennium B.C. Yin-Yang(陰陽) school related the cycle with the sixty four hexagrams, and the system of five notes(五音) and twelve pitch-pipes(十二律). The processes to relate them are called NaJia(納甲) and NaYin(納音) respectively and quoted in Shen Kuo(沈括, 1031-1095)'s MengQi BiTan(夢溪筆談, 1095). Yang Hui(楊輝) mathematicized the latter process in his book Xugu Zhaiqi Suanfa(續古摘奇算法, 1275). In this paper we show that Yang Hui introduced the earliest concept and notion of functions, in particular those with finite domains and codomains and then using the theory of congruences and composite of functions, he represented the NaYin process as a 1-1 onto function on the set of sexagesimal cycle into the set of pairs given by the five notes and twelve pitch-pipes.

Keywords: five notes, twelve pitch-pipes, sexagesimal cycle, NaYin(納音), Shen Kuo(沈括), MengQi BiTan(夢溪筆談, 1095), Yang Hui(楊輝), Xugu Zhaiqi Suanfa(續古摘奇算法, 1275), functions, congruences.

1 Introduction and the traditional NaYin process

The sexagesimal cycle(干支) has been introduced as early as the Shang(商 = 殷) dynasty in the second millennium B.C. for the dating of days. It is made of combinations of ten heavenly stems(天干) and twelve earthly branches(地支). The heavenly stems consist of Jia(甲, 1), Yi(乙, 2), Bing(丙, 3), Ding(丁, 4), Wu(戊, 5), Ji(己, 6), Geng(庚, 7), Xin(辛, 8), Ren(壬, 9), and Gui(癸, 10), and the earthly branches of Zi(子, 1), Chou(丑, 2), Yin(寅, 3), Mao(卯, 4), Chen(辰, 5), Si(巳, 6), Wu(午, 7), Wei(未, 8), Shen(申, 9), You(酉, 10), Xu(戌, 11), and Hai(亥, 12), where numbers indicate the order in the system. The set of heavenly stems(earthly branches, resp.) will be denoted by H(E, resp.) which is equipotent to $\{1, 2, \ldots, 10\}$ ($\{1, 2, \ldots, 12\}$, resp.). Using these notations, one can figure out the hexagesimal cycle as

$$\{(m, n) \in \{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 12\} \mid m \equiv n \mod 2\}.$$

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Further its elements are called by juxtaposing two elements as JiaZi(甲子), YiChou(乙丑), . . ., GuiHai(癸亥), or (1, 1), (2, 2), ⋯, (10, 10), (1, 11), (2, 12), (3, 1), ⋯, (10, 12) (see Table 1). The set of the sexagesimal cycle will be denoted by S, which is clearly equipotent to $\{1, 2, 3, \ldots, 58, 59, 60\}$.

The sexagesimal cycle has played a very important role in human affairs such as astrology and almanacs, and the arts of divination(術數) up to present days. It is used to represent year(歲次), month(月建), day(日辰) and hour(時) and to relate hexagrams in YiJing(易經) whose process is called NaJia(納甲).

Two basic frames of Chinese scales were introduced in the third century B.C. One is five notes(五音) consisting of Gong(宮), Shang(商), Jue(角), Zhi(徵) and Yu(羽) and the other is twelve pitch-pipes(十二律) which are divided into two parts, namely six Lu(六律) and six Lu(六呂). The former consists of Huang Zhong(黃鐘), Da Cu(大簇), Gu Xian(姑洗), Rui Bin(蕤賓), Yi Ze(夷則), Wu Yi(無射) and the latter of Da Lu(大呂), Jia Zhong(夾鐘), Zhong Lu(仲呂), Lin Zhong(林鐘), Na Lu(南呂), Ying Zhong(應鐘)([2, 6]).

Yin-Yang(陰陽) school was formed in the Former Han dynasty(206 B.C. - 24 A.D.). Yin and Yang are two mutually complementary principles whose ceaseless interplay gives rise to all natural phenomena. Yang and Yin are represented by odd and even numbers respectively. Five Elements(五行) consist of wood(木), fire(火), earth(土), metal(金), water(水) which also correspond to Jupiter, Mars, Earth, Venus, and Mercury respectively and circulate in a fixed circulating order. The Yin-Yang school is based on correlation made between the Five Elements, the four compass points with the center(中央, 東, 西, 南, 北), the four seasons, the five notes, the twelve months, the twelve pitch-pipes and the sexagesimal cycle. Indeed, water, fire, wood, metal and earth were represented by 1(north, winter), 2(south, summer), 3(east, spring), 4(west, autumn) and 5(center, earth) but the fixed circulating order of their force(氣) is given by wood(3), fire(2), earth(5), metal(4) and water(1)([2]).

The five notes Gong(宮), Shang(商), Jue(角), Zhi(徵) and Yu(羽) were corresponded to earth(5), metal(4), wood(3), fire(2) and water(1), respectively. The six Lu(六律) Huang Zhong(黃鐘), Da Cu(大簇), Gu Xian(姑洗), Rui Bin(蕤賓), Yi Ze(夷則), Wu Yi(無射) are conceived of as Yang and hence represented by 1, 3, 5, 7, 9 and 11 respectively and the six Lu(六呂) Da Lu(大呂), Jia Zhong(夾鐘), Zhong Lu(仲呂), Lin Zhong(林鐘), Na Lu(南呂), Ying Zhong(應鐘) as Yin and hence represented by 2, 4, 6, 8, 10 and 12 respectively, which determine the circulatory order of twelve pitch-pipes.

In the following, the sets of five notes and twelve pitch-pipes will be denoted by N and P respectively and then the set $N \times P$ has 60 elements and hence is equipotent to the set S of the sexagesimal cycle. Yin-Yang school made a 1-1 correspondence between two sets which is called NaYin(納音). Shen Kuo(沈括, 1031-1095) quoted the process in his book MengQi BiTan(夢溪筆談, [6]). Since the set E of earthly branches and P of twelve pitch-pipes have twelve elements, one has a 1-1 correspondence which may be recognized by the identity function of $\{1, 2, \ldots, 11, 12\}$, where the elements of E and P are as above represented by the elements of $\{1, 2, \ldots, 11, 12\}$. In other words, Zi $\mapsto$ Huang Zhong, Chou $\mapsto$ Da Lu, Yin $\mapsto$ Da Cu, ⋯, Xu $\mapsto$ Wu Yi, Hai $\mapsto$ Ying Zhong. Thus it remains to construct a function on N to the sexagesimal cycle which leads to a 1-1 correspondence between $N \times P$ and S of the sexagesimal cycle. The Na Yin process was designed by four rules, namely counterclockwise direction(左旋), pairing of yin and yang(娶妻), interval given by eight(隔八生子) and jump by three

甲子, 乙丑	丙寅, 丁卯	戊辰, 己巳	庚午, 辛未	壬申, 癸酉
商	徵	角	宮	商
甲戌, 乙亥	丙子, 丁丑	戊寅, 己卯	庚辰, 辛巳	壬午, 癸未
徵	羽	宮	商	角
甲申, 乙酉	丙戌, 丁亥	戊子, 己丑	庚寅, 辛卯	壬辰, 癸巳
羽	宮	徵	角	羽
甲午, 乙未	丙申, 丁酉	戊戌, 己亥	庚子, 辛丑	壬寅, 癸卯
商	徵	角	宮	商
甲辰, 乙巳	丙午, 丁未	戊申, 己酉	庚戌, 辛亥	壬子, 癸丑
徵	羽	宮	商	角
甲寅, 乙卯	丙辰, 丁巳	戊午, 己未	庚申, 辛酉	壬戌, 癸亥
羽	宮	徵	角	羽

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intervals(遁甲三元). Indeed the law of counterclockwise direction means that the circulating order is given by metal(4, Shang), fire(2, Zhi), wood(3, Jue), water(1, Yu) and earth(5, Gong) contrary to the order 3, 2, 5, 4, 1 of the force. The law of pairing of yin and yang is to associate the same note to each pair of consecutive heavenly stems starting from Jia. The law of interval given by eight is that two elements of the sexagesimal cycle with the difference 8 in their order correspond to the same note. Finally the law of jump by three intervals means that the last two laws are applied only to three intervals and then the next cycle begins anew by the law of counterclockwise direction. Thus JiaZi(甲子), YiChou(乙丑) $\mapsto$ Shang, i.e., (1, 1), (2,2) $\mapsto$ Shang by the first two laws and RenShen(壬申, (9, 9)), GuiYou(癸酉, (10, 10)), GengChen(庚辰, (7, 5)), XinSi(辛巳, (8, 6)) $\mapsto$ Shang as well by the law of interval given by eight. Furthermore, WuZi(戊子, (5, 1)), JiChou(己丑, (6, 2)) $\mapsto$ Zhi by the law of jump and then continue the same process to the rest of the sexagesimal cycle. These processes assign five notes to the half of the sexagesimal cycle. For the remaining half, one applies again the four rules from JiaWu(甲午, (1, 7)) and then one has a function $f : S \to N$. Values of f are indicated in Table 1.

In order to indicate the circulating order in the sexagesimal cycle, we use the quotient functions given by congruences as follows. The set of integers will be denoted by $\mathbb{Z}$. For $a, b, m \in \mathbb{Z}$, we say that a is congruent to b modulo m if m divides $a - b$ and write $a \equiv b \mod m$. It is well known that the relation given by the above relation is a congruence relation on the ring $\mathbb{Z}$ ([1]), the quotient ring is denoted by $\mathbb{Z}/[m]$ and the quotient function by $q : \mathbb{Z} \to \mathbb{Z}/[m]$.

Using these notions, the sexagesimal cycle is equipotent to $\mathbb{Z}/[60]$ so that these sets will be also considered as the set S of the sexagesimal cycle. Similarly the set E of twelve earthly branches and the set P of twelve pitch-pipes are identified with the quotient set $\mathbb{Z}/[12]$ and the set N of the five notes with $\mathbb{Z}/[5]$.

Let $g : \{1, 2, \ldots, 59, 60\} \to \mathbb{Z}/[12]$ denote the restriction of the quotient function $q : \mathbb{Z} \to \mathbb{Z}/[12]$. We now define a function $h = f \sqcap g : \{1, 2, \ldots, 59, 60\} \to N \times P$, i.e., $h(n) = (f(n), g(n))$ for $n \in \{1, 2, \ldots, 59, 60\}$, where the function f is given in the above. One can prove that the function h is indeed 1 - 1 onto and hence gives rise to the NaYin process. The detail of proof can be found in [11].

In all, the NaYin process by YinYang school has been derived by the above four rules. There are numerous explanations for the motivation of the above four rules and they are quite remarkable but not self-evident.

The purpose of this paper is to illustrate that Yang Hui(楊輝) mathematicized the traditional NaYin process in his book Xugu Zhaiqi Suanfa(續古摘奇算法, 1275). To do so, he introduced the earliest concept of functions, in particular those with finite domains and codomains, and that of composite of functions.

2 Yang Hui's NaYin QiLi(納音 起例)

Yang Hui is one of the four great mathematicians in Song(宋, 960-1279) and Yuan(元, 1271-1368) dynasties. The others are Li Ye(李冶, 1192-1279), Qin Jiu Shao(秦九韶, 1202-1261) and Zhu Shi Jie(朱世傑)([5, 7]). Yang Hui wrote XiangJie JiuZhang SuanFa(詳解九章算法, 1261) and Yang Hui SuanFa(楊輝算法, 1274-1275). The latter consists of three books ChengChu TongBian SuanBao(乘除通變算寶, 1274), TianMu BiLei ChengChu JieFa(田畝比類乘除捷法, 1275) and Xugu Zhaiqi Suanfa(續古摘奇算法, 1275)([9, 10]). In these books, Yang Hui included his own mathematical results together with the history of the theory of equations. The theory of equations, in particular the theory of solving polynomial equations formulated in the 11 ∼ 12th century, is one of the most important results in the history of Chinese mathematics. But the original books with the results are all lost and hence we can know about their history only through Yang Hui's works. Yang Hui is also an innovator. He didn't just follow traditional mathematics in various subjects but improved them by mathematical structures. It is well known that the diagrams HeTu(河圖) and LuoShu(洛書) played a very important role for Chinese philosophy, in particular in YiJing and Yin-Yang school. Traditionally LuoShu is known to be a magic square of order 3 and HeTu deals with numbers from 1 to 10. The labeling of these diagrams were strongly defended by Shao Yong(邵雍, 1011-1077) and Zhu Xi(朱熹, 1130-1200) who formed a mainstream of philosophy in Song dynasty([8]). Dissenting from the mainstream, Yang Hui labeled both diagrams in Xugu Zhaiqi Suanfa following Liu Mu(劉牧, 1011-1064)'s opinion that HeTu is the magic square. However, in this paper we will follow the traditional labeling, so that LuoShu is a magic square. Illustrating how to construct LuoShu, Yang Hui mathematicized it. Further, he added magic squares of order $n(4 \leq n \leq 10)$ and some variations of magic squares. His magic square of order 10 is not exact and corrected by Chosun mathematician Hong Jung Ha(洪正夏, 1684-?) in his GuIlJib(九一集, 1724, [4]).

In this section, we investigate Yang Hui's mathematicized NaYin process. For his theory, we use the following notions: The set H of heavenly stems means $\mathbb{Z}/[10]$, the set E of earthly branches $\mathbb{Z}/[12]$ and the equivalence class $[n]$ of n is simply denoted by n. Thus $\mathbb{Z}/[10] = \{1, 2, \ldots, 9, 10\}$ and $\mathbb{Z}/[12] = \{1, 2, \ldots, 11, 12\}$. But operations in these sets mean those in the quotient rings. For example, for $5, 8$, $5 + 8 = 3$ when they are elements of H, but $5 + 8 = 1$ for elements of E. The set S of sexagesimal cycle means the subset $\{(m, n) \mid 1 \leq m \leq 10, 1 \leq n \leq 12, m \equiv n \mod 2\}$ of the product set $H \times E = \mathbb{Z}/[10] \times \mathbb{Z}/[12]$.

Yang Hui defines a function $c : \{1, 2, \ldots, 10\} \to \{9, 8, 7, 6, 5\}$ for heavenly stems as follows:

$$c(1) = c(6) = 9, \quad c(2) = c(7) = 8, \quad c(3) = c(8) = 7,$$
$$c(4) = c(9) = 6, \quad c(5) = c(10) = 5. \quad \text{(See Figure 1)}$$

Proposition A. The function c for heavenly stems satisfies the following.

i) If $m \equiv n \mod 5$, $c(n) = c(m)$.

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ii) $c(n) + n \equiv 0 \mod 5$.

iii) $c(n+8) - c(n) \equiv -3 \equiv 2 \mod 5$.

The statements i) and ii) are immediate from the definiton of c. Since $c(n) \equiv -n \mod 5$ by ii), one has iii), for

$$c(n+8) - c(n) \equiv c(n+3) - c(n) \equiv -n - 3 - (-n) \equiv -3 \mod 5.$$

Yang Hui also introduced a function $e : \{1, 2, \ldots, 12\} \to \{9, 8, 7, 6, 5, 4\}$ for earthly branches as follows:

$$e(1) = e(7) = 9, \quad e(2) = e(8) = 8, \quad e(3) = e(9) = 7,$$
$$e(4) = e(10) = 6, \quad e(5) = e(11) = 5, \quad e(6) = e(12) = 4. \quad \text{(see Figure 1)}$$

Proposition B. The function e for earthly branches has the following properties:

i) If $m \equiv n \mod 6$, $e(n) = e(m)$.

ii) $e(n) + n \equiv 10 \equiv 4 \equiv -2 \mod 6$.

iii) $e(n+8) - e(n) \equiv -2 \equiv 4 \mod 6$.

The statements i), ii) are clear by the definition of e. Since $e(n) \equiv -n + 4 \mod 6$, one has

$$e(n+8) - e(n) \equiv e(n+2) - e(n) \equiv -n - 2 + 4 - (-n+4) \equiv -2 \mod 6.$$

We note that the above two functions have finite domains and codomains and that Yang Hui's concept of such functions are the earliest one in the history of mathemtics. He also introduced a very much descriptive method to represent the functions as in Figure 1, which may be used even now.

The function c for heavenly stems relates to a doctrine of YinYang school and extending this, Yang Hui defines mathematically the function e for earthly branches.

We now discuss values of functions c, e according to the two laws of interval given by eight and jump by three intervals and show that Yang Hui's functions reveal mathematical structures of two laws. Beginning at JiaZi(甲子), we calculate the values of the function c for heavenly stems and e for earthly branches by the above Proposition A, B which are as follows:

$$\begin{aligned} c(n) &: 9,6,8; \quad 5,7,9; \quad 6,8,5; \quad 7,9,6; \quad 8,5,7. \\ e(n) &: 9,7,5; \quad 9,7,5; \quad 9,7,5; \quad 9,7,5; \quad 9,7,5. \end{aligned}$$

Similarly, beginning at YiChou(乙丑), one has the values of c, e for the law of pairing of yin and yang as follows:

$$\begin{aligned} c(n) &: 8,5,7; \quad 9,6,8; \quad 5,7,9; \quad 6,8,5; \quad 7,9,6. \\ e(n) &: 8,6,4; \quad 8,6,4; \quad 8,6,4; \quad 8,6,4; \quad 8,6,4. \end{aligned}$$

We recall that the NaYin process in the previous section starts at JiaZi$((1,1))$ and JiaWu$((1,7))$. Since their heavenly stems are both 1, the values of c in the above table do not change when we begin with JiaWu instead of JiaZi. Furthermore noting that $1 \equiv 7 \mod 6$ for Zi(1) and Wu(7), the values of e in the above table are those of e when we begin at JiaWu.

Similarly, starting at the YiWei instead of YiChou, one has the exactly same table as above for values of c, e.

We note that the NaYin process is determined by a function on S onto the set $\mathbb{Z}/[5]$ of the five notes and hence the values of the function are devised modulo 5. Contrary to the fact that the values of c follow modulo 5 by Proposition A, the values of e follow modulo 6 by Proposition B. Thus the values of e in one cycle determined by the law of jump by three intervals, decrease by 2 but the next cycle begins increased by 4. Although $-2 \equiv 4 \mod 6$, they are not congruent modulo 5.

Yang Hui noted that the values of c increase by 2 and those of e decrease by 2 *in one cycle* and hence the sum of values of two functions c, e are fixed *in one cycle*. Indeed, the sums of values of two functions are 3, 4, 5, 1, 2 in the five cycles beginning at JiaZi. Similarly, the sums of values of two functions are 1, 2, 3, 4, 5 in the five cycles beginning at YiChou which is the pair of JiaZi. In all, in each cycle of the five cycles, the sum of sums of values of two functions at the pairs by the law of pairing of yin and yang is still fixed and the sums are 4, 1, 3, 5, 2 modulo 5 in the five cycles. We denote five cycles by I(JiaZi, RenShen, GengChen), II(WuZi, BingShen, JiaChen), III(RenZi, GengShen, WuChen), IV(BingZi, JiaShen, RenChen), V(GengZi, WuShen, BingChen) and the cycles determined by the pairs of the above cycles by I′, II′, III′, IV′, V′. Using these notions and collecting the above facts, we now have the following theorem.

Theorem. The functions c, e satisfy the following.

i) For the cycles I, II, III, IV, V determined by the laws of the interval given by eight(隔八生子) and jump by three intervals(遁甲三元), $c(m) + e(n)$ is fixed modulo 5 for any member (m, n) in the

same cycle of the five cycles. Furthermore, the sums are 3, 4, 5, 1, 2 for the cycles I, II, III, IV, V, respectively.

ii) For the cycles I′, II′, III′, IV′, V′, $c(m)+e(n)$ is fixed modulo 5 for any member (m,n) in the same cycle of the five cycles. Furthermore, the sums are 1, 2, 3, 4, 5 for the cycles I′, II′, III′, IV′, V′, respectively.

iii) For the cycles I, II, III, IV, V, $\{(c(m)+e(n))+(c(m+1)+e(n+1))\}$ is fixed modulo 5 for any member (m,n) in the same cycle of the five cycles and its pair $(m+1,n+1)$. The sums are 4, 1, 3, 5, 2 for the cycles I, II, III, IV, V, respectively.

iv) The above i), ii) and iii) hold for the five cycles determined by the two laws beginning at JiaWu(甲午).

Using the above theorem, Yang Hui has obtained the NaYin process. Indeed take any pair (m,n) which appears at an odd place in the ordered set S of sexagesimal cycles endowed with the usual order and its next pair $(m+1,n+1)$ and then calculate the sum

$$(c(m)+e(n))+(c(m+1)+e(n+1)).$$

He then has a function $s:S\to N$ which corresponds one of the five notes Gong(宮, 5), Shang(商, 4), Jue(角, 3), Zhi(徵, 2) and Yu(羽, 1) by the sum to (m,n) and $(m+1,n+1)$ and then a 1-1 correspondence $y=s\sqcap g:S\to N\times P$, where g is the function introduced in the previous section. One can easily figure out $g(m,n)=$ the nth pitch-pipe. The detail of the proof for y being 1-1 and onto can be found in [11]. In all y is a new NaYin process which Yang Hui called a temporary process(借音).

For the traditional NaYin process, Yang Hui introduced a function $t:\{1,2,3,4,5\}\to\{1,2,3,4,5\}$ as follows:

$$t(1)=2,\ t(2)=5,\ t(3)=3,\ t(4)=4,\ t(5)=1.$$

For a motivation of the function t, Yang Hui stated the following:

"金木自有聲 用木三金四本數 水遇土而有聲 火遇水而有聲

土遇火煆則有聲 故火用水數一 水用土數五 土用火數二"

We recall that one has a 1-1 correspondence between the five elements and the five notes by water(1, Yu), fire(2, Zhi), wood(3, Jue), metal(4, Shang), earth(5, Gong). Yang Hui's comment states that the permutation t of $\{1,2,3,4,5\}$ is determined by properties of sounds of the five elements.

Yang Hui shows that the function $f:S\to N$ for the traditional NaYin process is precisely $t\circ s:S\to N$ and hence he can have the traditional NaYin process h.

The tradtitional NaYin process is constructed by the order structure of the sexagesimal cycle but Yang Hui's process by the algebraic structure of the pairs of heavenly stems and earthly branches. Furthermore, the traditional one needs to be applied twice starting from JiaZi and JiaWu. But Yang Hui's process enables us to directly get the NaYin for any member of the sexagesimal cycle. Indeed for any member of the sexagesimal cycle with the odd earthly branch, one has its NaYin by iii) of the above theorem and the permutation t and for one with the even earthly branch, one has its NaYin by that of the sexagesimal cycle which precedes the given one.

Although Yang Hui uses iii) of the above theorem for his NaYin process, one may also use i) of the theorem. In this case, the permutation t should be replaced by $t(1) = 1$, $t(2) = 5$, $t(3) = 4$, $t(4) = 2$, $t(5) = 3$ for the traditional process. The property ii) of the theorem also determines a NaYin process. They provide three different NaYin processes. These new processes might not have been accepted because the traditional one was strongly tied with the doctrine of Yin-Yang school. We note that the new ones are precisely determined by the mathematical structure of the sexagesimal cycle and that the doctrine of Yin-Yang school may contain ambiguous claims.

3 Conclusion

Chinese mathematics was developed by JiuZhang SuanShu(九章算術) as the western mathematics by Euclid's Elements. JiuZhang SuanShu was completed as a mathematics book by Liu Hui's commentary ([3, 9, 10]). Liu Hui's preface for the book begins by the following statement.

> 昔在庖犧氏始畫八卦 以通神明之德 以類萬物之精 作九九之術 以合六爻之變
>
> 暨于黃帝神而化之 引而伸之 于是建歷紀 協律呂 用稽道原
>
> 然後兩儀四象精微之氣可得而效焉 記稱隸首作數 其詳未之聞也
>
> 按周公制禮而有九數 九數之流 則 九章是矣.

In ancient China, Fu Xi(庖犧=伏羲) has introduced the eight trigrams(八卦) and mathematics(九九之術) and extending these, they developed astrology(曆法) and music(律呂) and then Zhou Gong(周公) propagated six ethical codes for the education of young generations. One of those codes is JiuShu(九數) which is an origin of JiuZhang(九章). This claim of Liu Hui's has become the most important motivation for the study of mathematics throughout the history of oriental mathematics.

Yang Hui of Southern Song not only transmitted very important mathematical results obtained in the 11～12th century but also improved the traditional Chinese mathematics including JiuZhang SuanShu based on the mathematics of Song era. His improvement is prominent especially in his Xugu Zhaiqi Suanfa(續古摘奇算法, 1275). He didn't just follow the mainstream advocated by Shao Yong and Zhu Xi and showed that LuoShu, which was believed as one of bases for ShangShuXue(象數學), is simply a magic square of order 3. Further he claimed that the NaYin process arranged by the Yin-Yang school and its structure are simply mathematical consequences. To do so, he introduced a concept of functions including their composites and using this together with the theory of congruences, Yang Hui constructed the NaYin process and revealed the mathematical structure involved in the process.

But his ingenious achievements were neither appreciated nor understood by the next generations so that the earliest concept of functions in the history of mathematics has been completely forgotten in the eastern mathematics.

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