CHOSUN MATHEMATICIAN HONG JUNG HA'S LEAST COMMON MULTIPLES

朝鮮 算學者 洪正夏의 最小公倍數

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ABSTRACT

Though greatest common divisors were introduced in the first chapter fang tian(方田) of JiuZhang SuanShu(九章算術), least common multiples were rather neglected in mathematics of eastern Asia. We investigate a method of finding least common multiples that the greatest mathematician Hong Jung Ha(洪正夏, 1684～?) in Chosun dynasty introduced in his book GuIlJib(九一集, 1724). He first noticed that for the greatest common divisor m and the least common multiple n of two natural numbers a, b, $n = a\frac{b}{m} = b\frac{a}{m}$ and $\frac{a}{m}, \frac{b}{m}$ are relatively prime. He then showed that for natural numbers $a_1, a_2, \ldots, a_n$, their greatest common divisor d and least common multiple l, $\frac{a_i}{d}(1 \le i \le n)$ are relatively prime and there are relatively prime numbers $c_i(1 \le i \le n)$ with $l = a_i c_i (1 \le i \le n)$. This is one of the most prominent mathematical results on Number Theory in Chosun dynasty. The purpose of this paper is to show the process for Hong Jung Ha to capture and reveal the mathematical structure related to greatest common divisors and least common multiples. We also discuss Hong Jung Ha's pedagogical attitude.

Keywords: Hong Jung Ha(洪正夏, 1684～?), GuIlJib(九一集, 1724), Number Theory, greatest common divisors, least common multiples.

1 Introduction

JiuZhang SuanShu(九章算術, [4, 5]) established an approach to mathematics in China and eastern countries including Chosun, laying out mathematical subjects and presentations. In the first chapter fang tian(方田), it restricts the study of mathematics to the field $\mathbb{Q}$ of rational numbers rather than to the field of real numbers. The theory is described by relevant word problems that start with jin you(今有), their answers and processes to get them. But one can easily find mathematical structures involved in sequences of related problems. The field $\mathbb{Q}$ of rational numbers is given by fractions, and for reduction of fractions greatest common divisors and the Euclidean algorithm to compute them are introduced in the chapter. But least common multiples are rather neglected in the eastern mathematics except in a few cases, e.g. JiuZhang SuanShu and Zhang Qiu Jian SuanJing(張丘建算經,

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[4, 5]) to get common denominators([3]). Besides these, least common multiples were mostly restricted to relatively prime cases and thus obtained by multiplying the given numbers. General cases were dealt in Yang Hui(楊輝)'s XuGu ZhaiQi SuanFa(續古摘奇算法, 1275, [4, 5]) but Yang Hui didn't notice that for the greatest common divisor m and the least common multiple n of two natural numbers a, b, $ab = mn$ and he obtained common multiples instead of the least one. Eastern mathematicians did not show any interest in factorizations and hence prime factorizations except Yang Hui, who did use factorizations but only for short methods of multiplications and divisions in ChengChu TongBian SuanBao(乘除通變算寶, 1274, [4, 5]). Thus Number Theory of the eastern mathematics came to have a completely different history from the western counterpart.

The purpose of this paper is to show that Hong Jung Ha(홍정하, 洪正夏, 1684∼?) characterizes least common multiples in his mathematics book GuIlJib(구일집, 九一集, 1724, [2]), making a great contribution to Number Theory among others. Hong Jung Ha was from a Jung In(中人) family, passed the national examination for mathematicians(籌學取才) in 1706 and became Hoe Sa(會士, 從九品) in that year, Hun Do(訓導, 正九品) in 1718 and finally Gyo Su(敎授, 從六品) in 1720([9]). He wrote GuIlJib consisting of nine books. The first eight books contain 20 chapters with 493 problems, where a chapter(句股互隱門) in Book 5 deals with the right triangles with 78 problems and the last three books(開方各術門) are devoted to the theory of equations with 166 problems. Thus it may be assumed that his main interest in the book is the theory of equations. He studied Yang Hui SuanFa(楊輝算法 1274-1275, [4, 5]), Zhu Shi Jie(朱世傑)'s SuanXue QiMeng(算學啓蒙, 1299, [4, 5]), An Zhi Zhai(安止齋)'s XiangMing SuanFa(詳明算法, 1373, [4, 5]), Cheng Da Wei(程大位)'s SuanFa TongZong(算法統宗, 1592, [4, 5]) and Chosun mathematician Gyung Sun Jing(경선징, 慶善徵, 1616∼?)'s MukSaJib SanBup(묵사집산법, 默思集算法, [1]). Gyung is Hong's great grand uncle([9]). Using the TianYuanShu(天元術) in SuanXue QiMeng and the method of solving equations(開方法) in Yang Hui's TianMu BiLei ChengChu JieFa(田畝比類乘除捷法, 1275), Hong perfected the theory of equations. The first eight books were completed before 1713 and the final Book 9(雜錄) was added later in 1724. The final book is more of an appendix in which he discussed rudiments of astronomy, five notes(五音) and twelve pitch-pipes(十二律), and mathematical discussions with He Guo Zhu(何國柱) in 1713. He Guo Zhu was surprised at the excellence of Hong's mathematics, in particular at the TianYuanShu of which he presumably informed Mei Jue Cheng(梅瑴成, 1681∼1763).

Least common multiples are dealt in the first chapter GuiChun ChaBunMun(귀천차분문, 貴賤差分門) in Book 2. Begining with the least common multiple of two numbers, he arrives at the conclusion that for natural numbers $a_1, a_2, \ldots, a_n$, their greatest common divisor d and least common multiple l, $\dfrac{a_i}{d}(1 \leq i \leq n)$ are relatively prime and there are relatively prime numbers $c_i(1 \leq i \leq n)$ with $l = a_i c_i (1 \leq i \leq n)$. Hong Jung Ha's result reveals the mathematical structure of greatest common divisors and least common multiples.

2 Hong Jung Ha's least common multiples

We first quote the problem 7 in the chapter GuiChun ChaBunMun in Book 2.

今有甲乙二人同起程 只云甲日行八十五里 乙日行六十五里

問甲乙所行各幾日 以行里適等

答曰 甲一十三日 行一千一百五里, 乙一十七日 行一千一百五里

法曰 置甲日行八十五里 乙日行六十五里 約之得五爲法

另列八十五里以法五除之 得十七日此乙之日也

又列六十五里以法五除之 得十三日此甲之日也

乃列各日以其日行乘之 各得其里數一千一百五里 合問

一法 置甲日行八十五里 乙日行六十五里 乘之得五千五百二十五里爲實

另甲乙日行約之得五爲法 除之亦得各其行一千一百五里

The problem deals with the least common multiple of 85 and 65. Indeed, if two persons, A and B travel daily 85 Li(里) and 65 Li respectively, how many days do A and B take to travel the same Li of distance? First he finds the greatest common divisor 5 of 85 and 65 and by $\frac{85}{5} = 17$, $\frac{65}{5} = 13$ he concludes that A, B take 13 and 17 days respectively to cover the same distance of 1105 Li, i.e., the least common multiple is $85 \times 13 = 65 \times 17 = 1105$. He gives another method to get the least common multiple by $\frac{85 \times 65}{5} = 1105$, which means $5 \times 1105 = 85 \times 65$.

In the above problem, Hong precisely reveals the relation $dl = ab$ for the greatest common divisor d and the least common multiple l of a, b. Further using this in Book 9, he corrects Yang Hui's solution of a problem in XuGu ZhaiQi SuanFa. Indeed, he finds the least common multiple 84 of 12 and 28 and presents a common multiple 168 given by Yang Hui as another solution. He adds another problem to get the least common multiple 420 of 60 and 28 by the same relation in Book 9.

The next problem 8 is as follows:

甲乙丙三人行步不等 甲日行四十五里 乙日行四十里 丙日行三十五里

問幾日 以所行里相等

答曰 甲五十六日 行二千五百二十里 乙六十三日 行二千五百二十里

丙七十二日 行二千五百二十里

法曰列三人日行數互相約之得五 以除四十五里得九爲甲分母

以除四十里得八爲乙分母 以除三十五里得七爲丙分母

乃列於左行 又列各人日行於右行 以左行互乘右行三位 各得二千五百二十里 此相等之數 各以日行除之得各日

The problem 8 is exactly same as the above except that three persons, not two, travel daily 45 Li, 40 Li and 35 Li respectively. As above, he finds the greatest common divisor 5 and divides 45, 40 and 35 by 5 to get 9, 8, 7 which are called each person's denominator. He now applies the method HoSeung(호승, 互乘) to reduce the common denominator and then gets the least common multiple 2520 by $45 \times 8 \times 7 = 40 \times 9 \times 7 = 35 \times 9 \times 8 = 2520$. Further dividing 2520 by daily traveling Li, he obtains the number of days for each person to travel 2520 Li.

The last problem 9 dealing with least common multiple is as follows:

今有欲買牛馬騾驢 而其牛隻價十八兩 馬隻價十二兩 騾隻價九兩

驢隻價六兩 問四色幾隻之價適等

答曰 牛二 馬三 騾四 驢六 各價皆三十六兩

法曰各列四色價互相約之得三爲法 以除各價得牛六馬四騾三驢二

乃列於左行 又列各價於右行 以左行互乘右行四位

各得四百三十二兩爲實 各以其價除之得 牛二十四馬三十六

騾四十八驢七十二此亦等數

又四位互相約之–不能約者置之–得十二乃爲法

以除牛二十四得牛二 以除馬三十六得馬三 以除騾四十八得騾四

以除驢七十二得驢六

以除四百三十二兩得四色等價三十六兩 合問

A person wants to buy cows, horses, mules and donkeys priced at 18, 12, 9 and 6 coins respectively but to pay the same amount of coins for each animal. Thus the problem is about finding the least common multiple of 18, 12, 9 and 6. As in the problem 8, Hong gets 6, 4, 3 and 2 by dividing 18, 12, 9 and 6 by their greatest common divisor 3 and then using HoSeung(互乘) $18 \times (4 \times 3 \times 2) = 12 \times (6 \times 3 \times 2) = 9 \times (6 \times 4 \times 2) = 6 \times (6 \times 4 \times 3) = 432$, he obtains a common multiple 432 of the four numbers. Further he calculates the greatest common divisor 12 of the four numbers $4 \times 3 \times 2$, $6 \times 3 \times 2$, $6 \times 4 \times 2$, $6 \times 4 \times 3$ and gets 36, 2, 3, 4, 6 by dividing 432 and the above four numbers by 12. Clearly 2, 3, 4, 6 are relatively prime and Hong obtains the least common multiple $36 = 18 \times 2 = 12 \times 3 = 9 \times 4 = 6 \times 6$ of 18, 12, 9, 6, where 2, 3, 4, and 6 are the numbers of cows, horses, mules and donkeys respectively. In the process of the division by the greatest common divisor 12, he adds a comment that one doesn't need the process if the numbers are relatively prime as in the previous two problems.

Putting together the process of solving the above three problems, one can conclude the following:
the least common multiple of 85, 65 is given by $85 \times 13 = 65 \times 17 = 1105$,
the least common multiple of 45, 40, 35 is $45 \times 56 = 40 \times 63 = 35 \times 72 = 2520$, and
the least common multiple of 18, 12, 9, 6 is $18 \times 2 = 12 \times 3 = 9 \times 4 = 6 \times 6 = 36$, where the two numbers 13, 17, three numbers 56, 63, 72, and four numbers 2, 3, 4, 6 are all relatively prime.

In all, using the greatest common divisor and HoSeung(互乘), he finds a common multiple of given numbers. He divides the common multiple by the given numbers and then concludes that the common multiple is the least common multiple of given numbers when the quotients are relatively prime as in the first two problems. Otherwise, dividing the common multiple by the greatest common divisor of the quotients, Hong obtains the least common multiple as in the last problem.

Generalizing the above process, we have the following theorem.

Theorem(홍정하(Hong Jung Ha)) Let $a_1, a_2, \ldots, a_n$ be natural numbers and d, l their greatest common divisor and least common multiple respectively. Then one has the following.

i) $\dfrac{a_1}{d}, \dfrac{a_2}{d} \cdots \dfrac{a_n}{d}$ are relatively prime.

ii) There are relatively prime $c_1, c_2, \ldots, c_n$ with $l = a_i c_i (1 \leq i \leq n)$ and the converse also holds.

First we denote the set of natural numbers by $\mathbb{N}$ and define an order $\ll$ on $\mathbb{N}$ as follows: $x \ll y$ iff x is a divisor of y, i.e., $x \mid y$. Then it is obvious that $(\mathbb{N}, \ll)$ is a partially ordered set. It is also well known that for $a, b \in \mathbb{N}$, the greatest common divisor d(the least common multiple l, resp.) is precisely the meet $a \wedge b$(the join $a \vee b$, resp.) in the ordered set $(\mathbb{N}, \ll)$ and hence it is a lattice. We now go back to the proof of the theorem.

Proof i) is trivial by the definition.

Let's prove ii). Suppose that l is the least common multiple of $a_1, a_2, \ldots, a_n$, then there are $c_i(1 \leq i \leq n)$ with $l = a_i c_i (1 \leq i \leq n)$. Let p be the greatest common divisor of $c_1, c_2, \ldots, c_n$, then there are $x_i \in \mathbb{N}$ with $c_i = p x_i (1 \leq i \leq n)$. Thus $\dfrac{l}{p} = a_i x_i (1 \leq i \leq n)$ imply that $\dfrac{l}{p}$ is a common multiple of $a_1, a_2, \ldots, a_n$, so that $p = 1$. In all, $c_1, c_2, \ldots, c_n$ are relatively prime.

For the converse, assume that there are relatively prime $c_1, c_2, \ldots, c_n$ with $l = a_i c_i (1 \leq i \leq n)$. Let m be the least common multiple of $a_1, a_2, \ldots, a_n$. Then by the above proof, there are relatively prime $p_1, p_2, \ldots, p_n$ with $m = a_i p_i (1 \leq i \leq n)$. Since l is a common multiple of $a_1, a_2, \ldots, a_n$, there is q with $l = mq$, which implies $a_i c_i = a_i p_i q$. Thus $c_i = p_i q (1 \leq i \leq n)$ and therefore q is a divisor of relatively prime $c_1, c_2, \ldots, c_n$ and hence $q = 1$. This shows that $l = m$ is the least common multiple of $a_1, a_2, \ldots, a_n$.

3 Conclusion

In many occasions in GuIlJib, we can find that Hong Jung Ha did not simply follow the old methods of other mathematicians but tried to disclose mathematical structures in the problems and then established his own mathematics([6, 7, 8]). For example, in the introductory remark(凡例) of the book, he showed that Jia Xian(賈憲)'s triangle can be obtained by the synthetic division process used in ZengChengKaiFangFa(增乘開方法). He didn't have any information on Yang Hui's comment on the process which was quoted in YongLei DaDian(永樂大典, 1407) and Wu Jing(吳敬)'s JiuZhang SuanFa BiLei DaQuan(九章算法比類大全, 1450) but he did figure out the process for himself. Further, he extended the process to get the binomial coefficients of $a(x + b)^n$ for any a, b and also included the binomial coefficients of $(x - 1)^n$ up to $n = 12$. We have not been able to confirm that the latter was obtained by the synthetic division process. But if it is the case, then Hong might be the first eastern mathematician to use the division for negative numbers. It is notable that Hong derived the above results from just a few examples of extractions of the nth roots and Liu Yi(劉益)'s method to solve quadratic equations in those books mentioned in the introduction.

Hong Jung Ha's result on Number Theory discussed in this paper also shows his attitude to mathematics. He could obtain the least common multiple of two numbers a, b by $ab = dl$, where d, l are the greatest common divisor and the least common multiple of a, b respectively. But he further noticed that $l = a\dfrac{b}{d} = b\dfrac{a}{d}$ and that $\dfrac{a}{d}, \dfrac{b}{d}$ are relatively prime and this fact can be extended to the case of any number of natural numbers so that he captures the mathematical structure involving greatest common divisors and least common multiples.

The process to reach the results also indicates Hong's pedagogical attitude. Indeed the first problem deals with a case of two numbers, the second one with the exactly same case with three numbers and the last one with general cases where he notes that the first two problems are simply special cases of the general theory.

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