# A COMPARATIVE STUDY ON FINDING VOLUME OF SPHERES BY LIU HUI (劉徽) AND ARCHIMEDES

## An Educational Perspective to Secondary School Students

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**ABSTRACT**

The formula of finding the volume of a sphere was discovered independently in China and Greece. It was well-known that Archimedes of Syracuse (287?–212? B.C.) discovered and proved the formula of finding the volume of a sphere by comparing spheres with right circular cones and circular cylinders. Credited to *LIU Hui*'s (劉徽，225?–295? A.D.) commentaries on *Jiu Zhang Suan Shu* (九章算術，or Nine Chapters on the Art of Mathematics), ZU Geng (祖暅，480–525 A.D.) also found out the same formula by comparing spheres to a special solid called *Mouhefanggai* (牟合方蓋), which is also known as Bicylinder Steinmetz Solid named after Charles Proteus Steinmetz in late 19th century. While understanding the formula of finding the volume of a sphere is included in Hong Kong's Mathematics Curriculum in Key Stage 3, i.e. junior secondary education, it is worthwhile to compare the mathematical methods used in the discoveries and study the possible impact to students' development in the ability to 'think mathematically'.

## 1 Introduction to Archimedes and Liu's work

In the scope of the history of mathematics, Archimedes and *LIU Hui* were two giants in Greek Heritage and Confucian Heritage cultures respectively. Though little of their lives were known, some of their works were remained. In this article, three classical texts, namely *The Method of Archimedes Treating of Mechanical Problems—To Eratosthenes and On the Sphere and Cylinder, Book 1* by Archimedes and *Liu's Commentaries to Jiu Zhang Suan Shu* (九章算術劉徽注) would be particularly mentioned and discussed.

### 1.1 Archimedes' works

Discovering the formula of finding the volume of a sphere was one of Archimedes' proudest achievements in mathematics that legend told that Archimedes requested that a diagram representing the relationship between the volume of a sphere and that of a cylinder be sculpted on his tomb stone. Up to date, two different proofs to the formula by Archimedes were remained in the mentioned texts.

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In the treatise to Dositheus in *On the Sphere and Cylinder, Book 1*, Archimedes demonstrated a rigorous proof in the standard of classical Greek geometry. Using the concept of *reducio ad absurdum* and the method of exhaustion, Archimedes deduced the volume of a sphere from its surface area.

On the other hand, in the rediscovered text by Heiburg in 1906 of *The Method of Archimedes Treating of Mechanical Problems—To Eratosthenes*, an innovative approach of using a lever system to derive the volume of a sphere directly from the volumes of a cone and a cylinder was introduced.

Sir Thomas L. Heath's edition in *The Works of Archimedes and The Method of Archimedes* were referred in this article in the discussion of Archimedes works.

### 1.2 Liu's works

His commentaries to *Jiu Zhang Suan Shu* were most of Liu's remaining works. However, it should be highlighted that *Jiu Zhang Suan Shu* was already a classical text to the contemporary of Liu. Containing 246 arithmetic application problems and the numerical solutions, and 202 *Shu's* (術，or algorithms), the year of compilation of *Jiu Zhang Suan Shu* could be traced as early the Han Dynasty in the first century A.D.

In the original text of *Jiu Zhang Suan Shu* only included the numerical solutions and the algorithms to the problems. Liu's commentaries were great treasures in the development of mathematics in Confucian Heritage Culture because his commentaries provided the mathematical reasons on how the algorithms could indeed solve the problems stated which were also the important sources for the successors to decrypt this major classical mathematical text in ancient China.

In Chapter 4 of *Jiu Zhang Suan Shu*, a problem was stated to find the diameter of a sphere given its volume. The algorithm in the book implied that there had been a formula in finding volume of a sphere. However, Liu pointed out that the formula was wrong and disproved it by a cleverly designed counter-example. Though he could not derive a correct formula, his contribution was followed by *ZU Geng* about 200 years later and completed the search of the formula of finding the volume of a sphere in ancient China.

In this article, *Liu's Commentaries to Jiu Zhang Suan Shu* referred to the edition in *Qin Ding Si Ku Quan Shu* (欽定四庫全書，or *Imperial Collection of Four Treasures*) compiled in Qin Dynasty in 18th century. In this edition, besides Liu's commentaries there were also additional remarks given by *LI Chunfeng* (李淳風，602 – 670 A.D.) who had recorded the completion of works by Zu.

## 2 The approaches to find the volume of a sphere

In the following section, mathematical details will be discussed briefly. The article did not intend to give the complete proofs, but some mathematical insights would be highlighted as the points for further discussion in Section 3.

### 2.1 Archimedes' 'mechanical proof'[1]

Stated as Proposition 2 in *The Method of Archimedes Treating of Mechanical Problems*, Archimedes first considered three solids: a sphere; a cone with height the same as the diameter of the sphere whereas

[1]Heath, T. (ed.), 1912, *The Method of Archimedes: A Supplement to the Works of Archimedes 1897*, Cambridge: the University Press, pp. 18–21

the diameter of the base doubled that of the sphere and a cylinder of same base and height to the cone.

Next, he compared the areas of the cross sections of the solids parallel to their bases. Cut the solids at the same distance from the top of the solids along the height. One could prove that the ratio of the area of the cross section of the cylinder to the sum of the area of the cross section of the cone and the sphere is equal to the ratio of the height of the solids and the distance that was fixed. Denote $A(X, h)$ as the area of the cross section of solid $X$ at distance $h$ from the top of the solids, one could derive the following:

$$A(Cylinder, h) \times h = [A(Cone, h) + A(Sphere, h)] \times 2R$$

where R is the radius of the sphere.

The most interesting part of this proof was that Archimedes treated the above expression in a way analogous to the weight (i.e. force) and the moment arm. Hence, he 'balanced' the weight of the cylinder to the cone and the sphere with an aid of a lever system. It should also be noted that the moment arm of the side of the cone and the sphere was fixed but that of the cylinder varied as the distance of the cut.

Lastly he treated all these circles as thin slides that would add up to the solids. With proper arrangement (the cone and the sphere placed vertically at the distance of $2R$ away from the pivot on one side and the cylinder placed horizontally along the lever from the pivot to the distance of $2R$ on the other side), one could find out the volume of the sphere by considering the position of the centre of gravity of the cylinder, which lied on the position which was at a distance of $R$ from the pivot.

### 2.2 Archimedes' 'formal proof'[2]

In *On the Sphere and Cylinder, Book 1*, Archimedes somehow extended his idea on finding the area of circle by comparing the area of the inscribed and circumscribed regular polygons. For the case of sphere, he compared the solid of revolution of the inscribed and circumscribed regular polygons.

Archimedes claimed that the volume of a sphere was four times of the volume of a cone with the base equal to the greatest circle and height equal to the radius of the sphere. To prove the claim, he constructed two regular polygons with sides of $4n$. One of which was inscribed in the greatest circle of the sphere whilst the other circumscribed the circle. With a suitably large $n$, one could control the ratio of the sides.

Moreover, with the similarity property of the solids of revolution of the polygons, one could acquire the ratio of volumes of the solids as the cube of the ratio of the sides. Archimedes then went on using *reducio ad absurdum* to remove the possibility that the volume of a sphere was greater or smaller than four times of the volume of a cone with the base equal to the greatest circle and height equal to the radius of the sphere.

Even though the proof was logically sounded, one would be aware that if Archimedes did not know the exact ratio between the volume of a sphere and that of a cone, the proof would have broken down because it would be impossible to choose the number $n$ to control the ratio of the sides. Hence, it was reasonable to believe that Archimedes did find out the method of finding the volume of a sphere somewhere before he worked out a formal proof in *On the Sphere and Cylinder, Book 1*.

[2]Heath, T. (ed.), 1897, *The Works of Archimedes*, Cambridge: the University Press, pp. 41–44.

## 2.3 Liu's work and Zu's completion[3]

In the last question in Chapter 4 entitled "*Shao Guang*" (少廣，or "*The Study of Unknown Breadth*"), it stated "Given a sphere with volume 4500 units, find the measure of the diameter" (今有積四千五百尺，問為立圓徑幾何). The provided algorithm to the solution was (1) Multiply $16/9$ to the volume (置積尺數以十六乘之九而一); and then (2) take the cubic root of the product (所得開立方除之即圓徑). In other words, the formula of finding the volume of a sphere could be rearranged as:

$$V = \frac{9}{16}d^3 = \frac{3}{2}\pi r^3$$

taking $\pi$ equal to 3, which was a common practice in ancient China.

Liu gave a possible reason on the formulation of the above relation by considering a sphere which was inscribed by a cylinder and the cylinder was inscribed by a cube. He believed his precedents took the cross sections at the greatest circle of the cube horizontally and vertically and compared the sphere to the cylinder and then the cylinder to the cube. The following table illustrated the shapes of the solids:

| | Horizontal cross section | Vertical cross section |
|---|---|---|
| Sphere | Circle | Circle |
| Cylinder | Circle | Circumscribed square |
| Cube | Circumscribed square | Circumscribed square |

Since the cross sections of the sphere and the cylinder were in one way the same and in the other way a circle to its circumscribed square, meaning that the ratios of cross section areas were $1 : 1$ horizontally and $\pi : 4$, or $3 : 4$, on vertically. Multiplying, the ratio of volume between a sphere and a cylinder was hence $3 : 4$. Similarly, the ratio of volume between a cylinder and a cube was also $3 : 4$. The product of these ratios deduced that the ratio of volume between a sphere and its circumscribed cube was $9 : 16$.

Liu was well aware the logical loopholes of this argument: the cross sections of the solids were not necessarily always touched each other but the cross section of a sphere must lie in the interior of that of the cylinder, and that of the cylinder must also lie in the interior of that of the cube. Hence, this method was an over-estimate of the volume of the sphere. But since approximating $\pi$ by 3 was an under-estimate, Liu concluded that the formula suggested was just accidentally a reasonable approximation (是以九與十六之率偶與實相近).

He went on to construct the special solid *Mouhefanggai* which circumscribed a sphere to demonstrate that the estimation was better in estimating *Mouhefanggai* than the corresponding sphere.

Given a cube with side length $d$, its inscribed *Mouhefanggai* was constructed by the perpendicular intersecting surface of two cylinders of diameter $d$, i.e. the segments of axes of the cylinders with end points at the bases were perpendicular bisectors to each other. Hence, the shape of all the cross sections of *Mouhefanggai* in the direction perpendicular to any one of the axes were all circles of diameter $d$ whereas that of the cube were all squares of side length $d$. Therefore, Liu claimed that the approximation method was more suitable for finding the volume of *Mouhefanggai* instead because the cross sections really touched each other everywhere along the axes. However, since a sphere was inscribed by *Mouhefanggai*, the approximation was an over-estimate to a sphere (而丸猶傷多耳).

[3] *LIU Hui, LI Chunfeng* (eds.) (劉徽注，李淳風注釋), *Qin Ding Si Ku Quan Shu: Jiu Zhang Suan Shu*.

Noted that if one considers all the cross sections of the sphere and *Mouhefanggai* in the plane parallel to the plane containing the axes of cylinders in the construction of *Mouhefanggai*, all the cross sections of *Mouhefanggai* were the circumscribed squares of the cross sections of the sphere which were circles everywhere along the axes. Hence, Liu pointed out that the exact ratio between the volume of a sphere to the corresponding *Mouhefanggai* was $\pi : 4$, or approximately $3 : 4$. However, Liu failed to discover the method of calculating the volume of *Mouhefanggai*.

It took two hundred years to completely solve the problem in Zu. Zu' works were almost lost. The only remaining results were remarked by Li in *Jiu Zhang Suan Shu*. Zu was recorded to be the earliest mathematician to state explicitly the *ZU's Axiom* (夫疊棊成立積，緣冪勢既同，則積不容異), or known as Cavalieri's Principle.

Instead of directly found the volume of *Mouhefanggai*, Zu picked out one-eighth of *Mouhefanggai* and the cube and studied the difference of the cross sectional areas of the cube and the part *Mouhefanggai* along the the plane parallel to the plane containing the axes of cylinders in the construction of *Mouhefanggai*. He figured out that the difference was equal to the cross section of an inverted square pyramid with base length and height of $d/2$ at the same height. Hence, he found out the volume of *Mouhefanggai* by subtracting the volume of pyramid from the cube and also completely discovered the precise formula on finding the volume of a sphere.

## 3 How do these related to students in secondary school?

Without any knowledge of calculus for students in Key Stage 3 in Hong Kong who need to understand and use the formula of finding the volume of a sphere[4], it is important for teachers to explore the means to account for the formula. It is thus reasonable to study how the formula was originated and how can such historical findings be integrated in students' learning. Three approaches have been discussed. In this section, some comparison will be made in terms of the cultivation of students' development of 'mathematical mind'.

One may begin with whether the discovery of the volume of a sphere cohered with the geometric intuition. Interestingly, Archimedes and Liu both adopted the use of ratio to represent the volume of a sphere, though the objects of comparison were different.

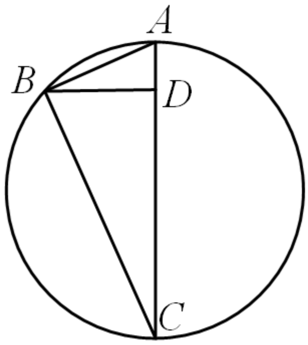
In Archimedes' 'formal proof', solids of revolution of the inscribed and circumscribed regular polygons were compared with the sphere. Such comparison is natural because it is visually well accepted that a circle can be infinitesimally approximated by a regular polygon. Most students might have been equipped with similar techniques on finding the area of a circle which they were usually taught to approximate the area of sectors by the area of triangles.

However, the major difference between the treatment of the area of a circle and the volume of a sphere is that it is easy to rearrange the triangles in a way that students could easily find the area by simple calculation, but finding the volume of the solid of revolution is somehow as difficult, if not more, as calculating the volume of the sphere directly to students. Indeed, Archimedes did not directly calculate the volume of the solid of revolution either. Instead, he planned to complete the proof with more than 20 propositions which pointed to transform the solid of revolution into an isosceles cone

[4]The Curriculum Development Council, 1999, *Syllabuses for Secondary Schools: Mathematics (Secondary 1–5)*, Hong Kong: The Education Department, p. 20.

of the same volume with known base and height. The procedure was far more complicated than the case of finding area of circles.

On the other hand, Archimedes reached the finish line on finding the volume of a sphere in a much faster way with the aid of a lever system. Not entirely mathematically, it is worthwhile for students to experience that one can 'borrow' the idea in the fields of studies other than in mathematics to solve mathematical problem. The starting point of formulating this approach might be a simple geometric observation that in the following figure, $\triangle ABC$ was similar to $\triangle ADB$:




where $ABC$ is a circle with diameter $AC$, and $BD$ perpendicular to $AC$. Then it is easy to come to the relation $AD^2 + BD^2 = AC \cdot AD$. And the most intelligent part was how Archimedes transformed the lengths stated in the relation to a physically sounded setting for the lever system. Though the transformation of the sides was quite artificial, it may give some insights for students to read through a basic geometric property to solve a more complicated situation.

Compared with Archimedes' approaches, Liu and Zu's approach relied on another geometric intuition, namely Zu's Axiom. Liu and Zu studied the sphere by the set of cross sections like Archimedes did in his 'mechanical proof'. The shape of *Mouhefanggai* was unfamiliar, but obviously Liu constructed it in the sense of refining an approximation. Knowing the limitation of the approximation given in *Jiu Zhang Suan Shu*, that a cube or a cylinder was too large to touch the sphere, he refined the shape of inscribing the sphere by intersecting another cylinder in another direction with the consideration that a cylinder touched a sphere at the two end points of the small circles other than the greatest circle. Hence, the intersections of two perpendicular cylinder touched the sphere as a square circumscribed a circle.

The idea of refinement was important in mathematical studies. Students in secondary schools might have an illusory thought that all mathematical problems could be solved directly starting from the very beginning, and if one could not solve the problem in such a way he/she failed in the whole problem. Students could be demonstrated with such an example that in the history of mathematics, mathematical breakthroughs were highly valued even though such breakthroughs may not solve the entire problem. Since most of the mathematical contents in the Mathematics Curriculum in Hong Kong are well developed mathematical facts and the emphasis on the process of development of mathematics is yet to be put, more accounts on the mathematical achievements in different stages in the history of mathematics could plant the humanity root back to students rather than considering mathematics as an artifact.

Zu's work demonstrated another advanced way of thinking. The idea of observing the exterior of

*Mouhefenggai* was another breakthrough in attacking the problem. It is also a vital technique that a student in mathematics should be well equipped with, and it is another way to transform a mathematical problem to another 'solvable' problem. By a wise application of *Gou-gu* Theorem (勾股定理), or Pythagoras Theorem, Zu related the difference of areas of two squares with a third one. Zu's work could serve as a demonstration on inter-relating various mathematical knowledge.

Educationally, the three approaches were both valued means in demonstrating how a mathematician attacked a mathematical problem with diverse emphasis. Studying across different approaches to the same problem by comparison and contrast could create a more comprehensive mathematical view. It is note-worthy that finding the volume of a sphere is just one of the remarkable examples in the history of mathematics. More similar comparative studies can be done to enhance students' development in 'thinking mathematically'.

## 4 Conclusion

What Archimedes, Liu and Zu had accomplished elegantly demonstrated the transition from intuition to abstract mathematical concept in different angles. The inclusion of these historical significance may interest students in mathematics to explore the development of mathematics in a humane and comprehensive way instead of learning mathematics through mechanical calculation. The mathematical ideas, the historical perspectives and the pedagogical measures should enhance each other in the long journey of developing a mathematical mind.

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